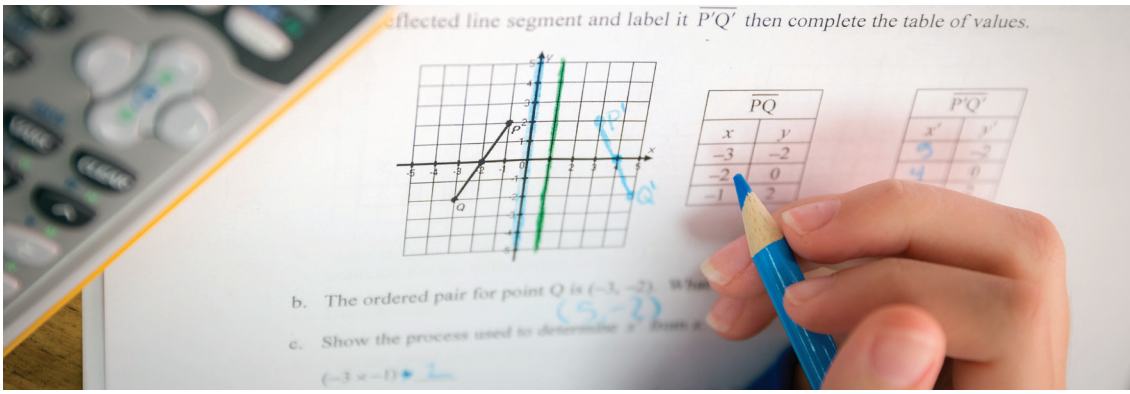
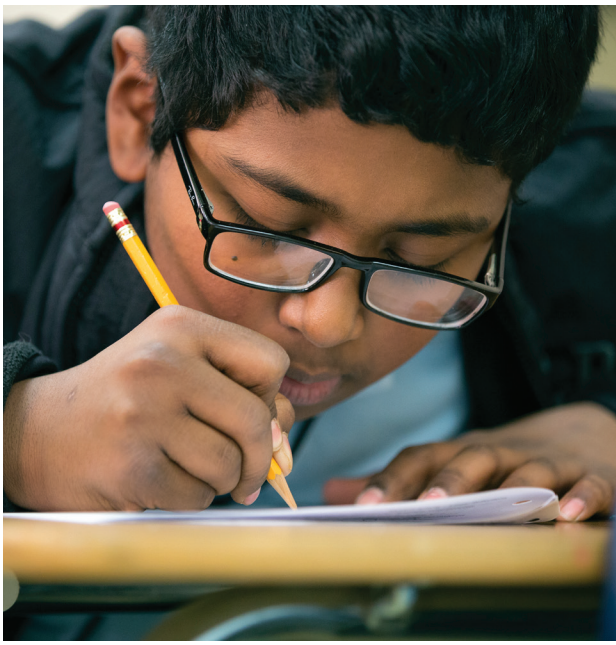
MODULE 4





MATHEMATICS

Analysis of Functions: Piecewise Graphs

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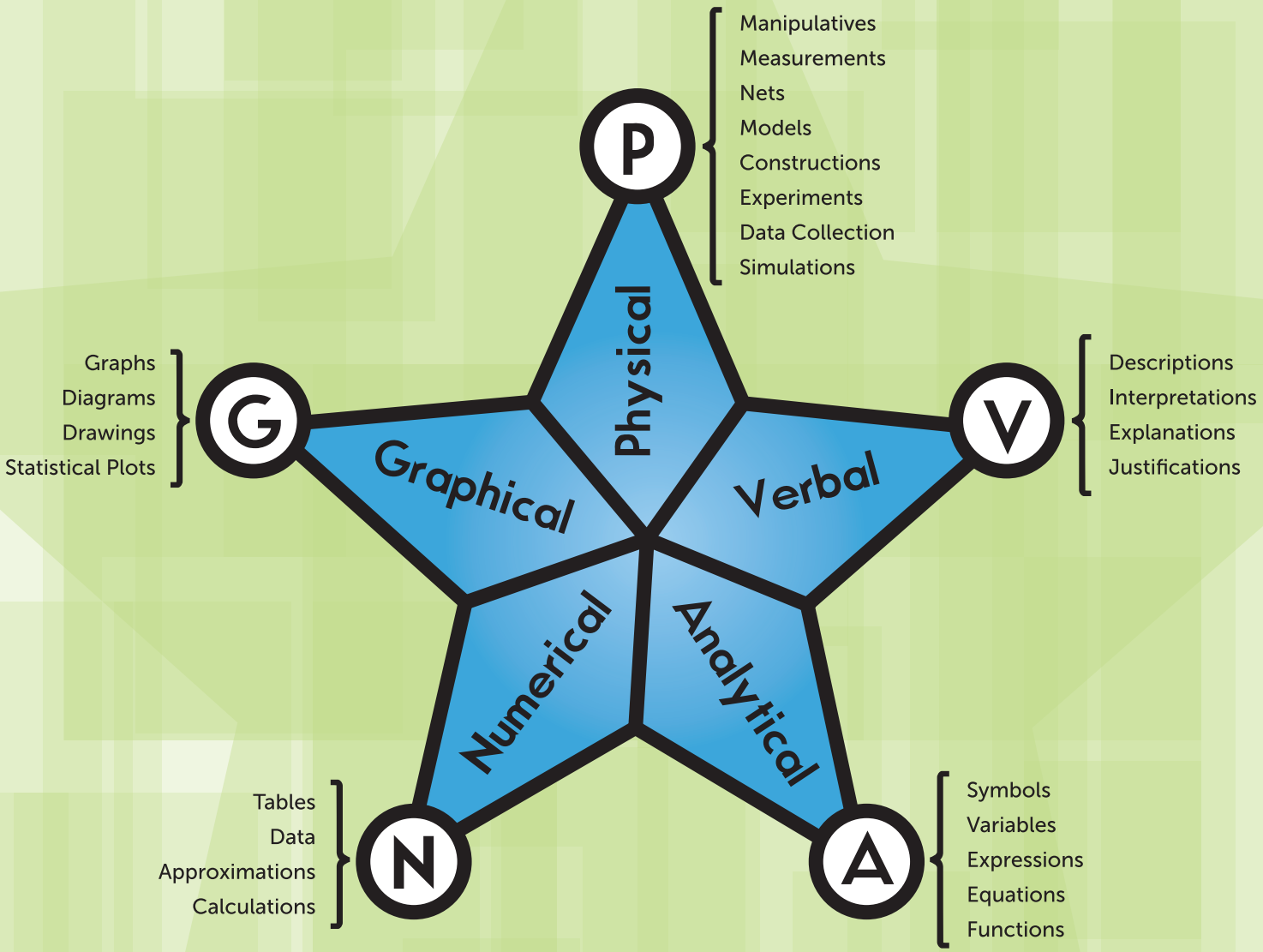
Using Multiple Representations in Mathematics



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MATHEMATICS

Module 4

Analysis of Functions:
Piecewise Graphs

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Analysis of Functions: Piecewise Graphs

CONTENTS

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Belief Statements

Accomplished, dynamic teachers are knowledgeable in their content and confident in their abilities to prepare students for higher education. They create classrooms in which students:

- engage intellectually to develop conceptual understanding
- generate their own ideas, questions, and propositions
- interact collegially with one another to solve problems
- employ appropriate resources for inquiry-based learning

Our teacher training program offers meaningful support to teachers as they construct these effective classrooms. Through tested content materials and research-based instructional strategies, our program enables and encourages them to:

- choose significant and worthwhile content and connect it to other knowledge
- use appropriate questioning strategies to develop conceptual understanding
- clarify to students the importance of abstract concepts and “big questions”
- use formative assessments to improve instruction and achieve higher goals
- guarantee equitable access for all students to information and achievement

Analysis of Functions: Piecewise Graphs

Middle Grades

MODULE DESCRIPTION

Middle school teachers examine how concepts involving analysis of functions progress from sixth grade to calculus. Training begins with a strategy for teaching students the difference between the path of a person's walk and a distance-time graph of the walk. Participants work through and discuss teaching strategies for classroom-ready middle grades lessons and assessments in which students interpret distance-time and rate-time graphs. In addition, they discuss selected questions from Algebra 1 and Geometry or Math 1 and Math 2 lessons on graphing and defining piecewise functions.

LEARNER OUTCOMES

Participants will:

- Compare expectations for students from sixth grade math through pre-calculus on the topic of analysis of piecewise functions to increase vertical alignment.
- Apply deeper content-based knowledge to increase instructional rigor in order to prepare students for high school math courses leading to college-level calculus in an AP class or university setting.
  - Interpret distance-time and rate-time graphs.
  - Model distance-time and rate-time graphs with role play.
  - Write equations of piecewise functions.
  - Analyze key features of piecewise functions.
- Identify instructional strategies that teachers can use to assist students in developing the habits of mind that are required for college and career readiness.

Analysis of Functions: Piecewise Graphs

High School

MODULE DESCRIPTION

High school teachers examine how concepts involving analysis of functions progress from sixth grade to calculus. Participants begin by reviewing how these concepts are introduced at the middle school level using distance-time and rate-time graphs. As the training progresses through the vertical strand, teachers work selected questions from and discuss teaching strategies for the high school lessons. In these activities, students write equations for, identify key features of, and analyze the effects of transformations on piecewise graphs.

LEARNER OUTCOMES

Participants will:

- Compare expectations of students from sixth grade math through pre-calculus on the topic of analysis of piecewise functions to increase vertical alignment.
- Apply deeper content-based knowledge to increase instructional rigor in order to prepare students for college-level calculus in an AP class or university setting.
    - Interpret distance-time and rate-time graphs.
    - Model distance-time and rate-time graphs with role play.
    - Write equations of piecewise functions.
    - Analyze key features of piecewise functions.
    - Apply transformations on specific portions of a piecewise function to make the function continuous.
- Identify instructional strategies that teachers can use to assist students in developing the habits of mind that are required for college and career readiness.

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Analysis of Functions:
Piecewise Graphs

LESSONS AND ASSESSMENTS

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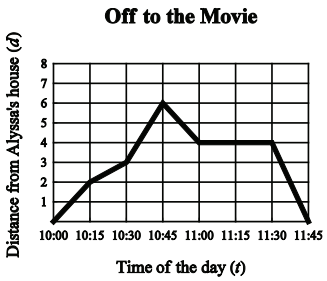
Analysis of Functions: Piecewise Graphs Content Progression Chart

6th Grade	7th Grade	Algebra 1	Geometry	Algebra 2	Pre-Calculus
Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/ or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/ or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/ or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	For a continuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.
Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world applications (including rate graphs).	Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world applications (including rate graphs).	Use ratios and/or equations to determine a value on a continuous piecewise function.	Use ratios and/or equations to determine a value on a continuous or discontinuous piecewise function.	Use ratios and/or equations to determine a value on a continuous or discontinuous piecewise function.
		Write a piecewise linear function based on a graph.	Write a piecewise function based on a graph (may include portions of a circle and linear pieces).	Write a piecewise function based on a graph (may include portions of a circle and/or linear and parabolic pieces).	Write a piecewise function based on a graph (may include portions of conic sections and/or linear, rational, or trigonometric pieces).
		Convert between absolute value functions and piecewise linear functions.	Convert between absolute value functions and piecewise linear functions.	Convert between absolute value functions and piecewise linear functions.	Convert between absolute value functions and piecewise linear function.
			Examine the effects of the transformation $af(x-c)+d$ on a continuous piecewise function including effects on area.	Examine the effects of the transformation $af(x-c)+d$ on a continuous or discontinuous piecewise function.	Examine the effects of the transformation $af(b(x-c))+d$ and $f(\|x\|)$ on a continuous or discontinuous piecewise function.
			Determine the slope at points on a piecewise function (may include portions of a circle and linear pieces).	Determine the slope at points on a piecewise function (may include portions of a circle and linear pieces).	Determine the slope at points on a piecewise function (may include portions of a circle and linear pieces).
			Given a point on a circle, determine the equation of a tangent line to a circle.	Given a point on a circle, determine the equation of a tangent line to a circle.	Given a point on a circle, determine the equation of a tangent line to a circle.
				Examine the discontinuity of a piecewise function as well as use transformations to make a function continuous.	Examine the discontinuity of a piecewise function as well as use transformations to make a function continuous.
					Determine the concavity of a continuous or discontinuous piecewise function.
					Determine $g(a)$, given g defined as a transformation of f.

Analysis of Functions: Piecewise Graphs Concept Development Chart

Grade 6

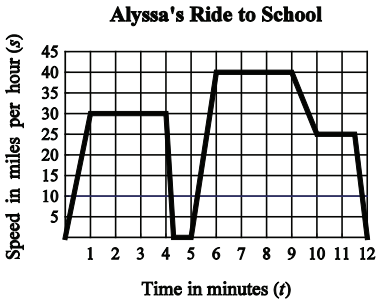
Alyssa and her older sister walked from their house to the movie theater. When they arrived, they realized they did not have enough money for the movie, so they decided to return home along the same path. On their way home, they decided they had enough money to purchase ice cream at Creamy Cones. After they finished their ice cream, they returned home along the same path.



Between what times did the girls stop at Creamy Cones?

Grade 7

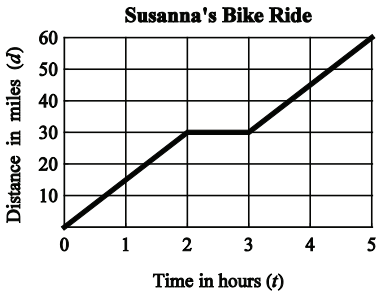
On Tuesday, Alyssa's mom allowed her older sister, Susana, to drive her to school. The graph models their speed in miles per hour as they traveled away from home.



Is their speed on the intervals $0 \le t \le 1$ and $5 \le t \le 6$ increasing or decreasing? On which of these intervals is their speed changing the fastest?

Algebra 1

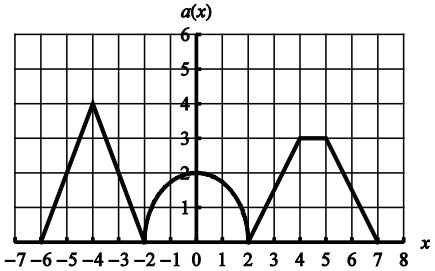
Susana is a member of a bicycle club. The graph shows the relationship between her time in hours and distance in miles for the first 5 hours of a bike ride.



What is the piecewise function for the graph on the time interval $0 \le t \le 5$?

Geometry

The graph of $y = a(x)$ is composed of a triangle on the interval $-6 \le x \le -2$, a semicircle on the interval $-2 \le x \le 2$, and a trapezoid on the interval $2 \le x \le 7$.



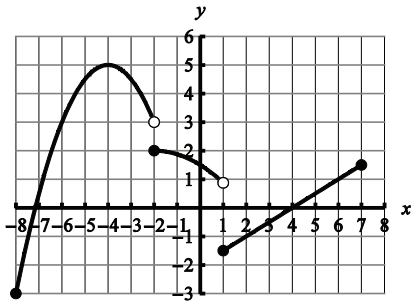
What is the equation of the piecewise function on the interval $-6 \le x \le 7$?

Which of the following transformations would increase the area bounded by the transformation of $a(x)$, the x-axis, $x = -6$, and $x = 7$ by the greatest amount?

I. $2a(x)$ II. $-a(x)$ III. $a(x) + 2$

Algebra 2

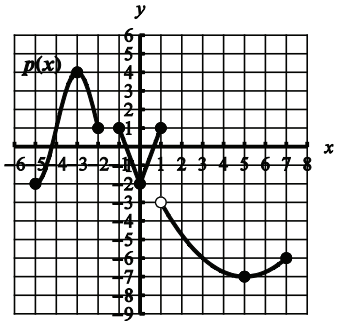
The piecewise function, $g(x)$, is composed of a parabola with its vertex at $(-4, 5)$ on the interval $[-8, -2)$, a parabola with its vertex at $(-2, 2)$ on the interval $[-2, 1)$, and a line on the interval $[1, 7]$.



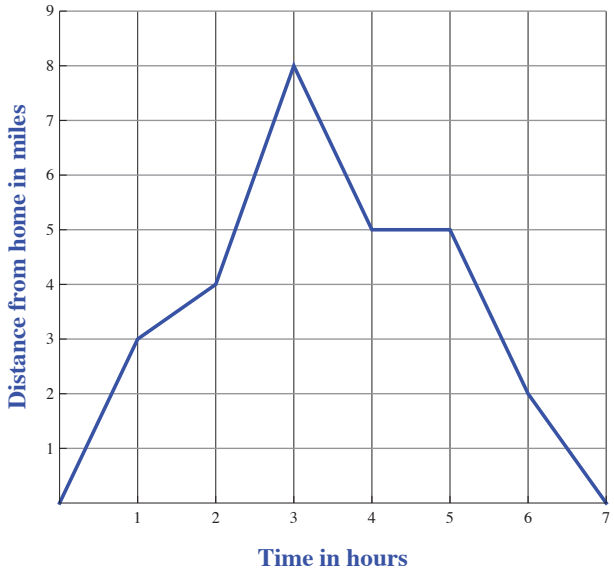
Describe two options for ways to translate one of the parabolas and the line to form a continuous function on $[-8, 7]$. Using one of your options, write the equation of the new continuous function, $h(x)$.

Pre-Calculus

The piecewise function, $p(x)$, is composed of a sinusoidal function on the interval $[-5, -2]$, an absolute value function on the interval $[-1, 1]$, and a parabola with vertex $(5, -7)$ on the interval $(1, 7]$.



Describe four options for ways to translate the three pieces to create a continuous function. Using a translation of the sinusoidal piece and the parabolic piece, write the equation of the new continuous function, $s(x)$.



Interpreting Distance Graphs

ABOUT THIS LESSON

This lesson presents students with a distance-time graph and asks them to read and interpret various characteristics of the graph in the context of a real-world scenario. By relating the graph to a table of time and distance data, students can draw conclusions about particular data points, time and distance intervals, and even speed. The Teaching Suggestions include scenarios to use with students prior to beginning the student activity pages to reinforce the idea that the path of a trip and the distance-time graph that results from graphing the trip are not the same.

This lesson focuses on reading, interpreting, and understanding a graph of time with respect to distance. The lesson reinforces using units in both the work and the answer of any given question and giving a specific reason to support an answer.

OBJECTIVES

Students will

- walk a path along a straight line following specific distance and time directions.
- create a graph of the path of a walk and compare it to a distance-time graph of the walk.
- analyze a distance-time graph for specific characteristics.
- use unit rates to determine the coordinates of a point not at a grid intersection.

LEVEL

Grade 6 or Grade 7 in a unit on coordinate graphing or rates

MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

TEACHER PAGES

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill.

Targeted Standards

6.EE.9: Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.*
See questions 5-6, 9-11

7.RP.2b: Recognize and represent proportional relationships between quantities.
(b) Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
See questions 5-6, 8, 11

Reinforced/Applied Standards

5.G.2: Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.
See questions 1-4, 7-8

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.6: Attend to precision.
Students use accurate units and correct mathematical language in their analysis of the graph.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Determine the coordinates of a point in Quadrant I
- Interpret a point in Quadrant I in terms of the units on the graph

ASSESSMENTS

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – 6th Grade Free Response Questions
- Analysis of Functions: Piecewise Graphs – 6th Grade Multiple Choice Questions
- Analysis of Functions: Piecewise Graphs – 7th Grade Free Response Questions
- Analysis of Functions: Piecewise Graphs – 7th Grade Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages

TEACHING SUGGESTIONS

To introduce the idea that the distance-time graph is not the path being walked, use the following scenarios before beginning the student activity pages:

Scenario 1: Ask a student to walk 8 feet across the front of the room, moving along a horizontal line, and then to return to the starting position. Ask the class to draw a picture of the student's path. The graph should be a horizontal line segment, 8 "feet" long.

Scenario 2: Ask a student to walk across the room as you read the following set of instructions. (These instructions model the walk described in the student activity pages.)

1. From your starting point, walk 3 feet moving at a constant rate.
2. Walk 1 foot more at a constant rate that is slower than your original rate.
3. Walking at a rate that is faster than your original rate, go forward 4 feet.
4. Turn around and walk 3 feet back toward your starting point at the original constant rate.
5. Stop to rest.
6. Walk 3 feet toward the starting point at the original constant rate.
7. Walk 2 feet to return to the original position at a slower constant rate.

Again, ask the class to draw a graph of the student's path. The graph should be exactly the same as the one drawn in Scenario 1. The distance-time graph for Ann's walk shown in the lesson indicates how far Ann is from home at a given time. The independent variable for the graph is time while her distance from home is measured along the vertical axis

On the interval, $5 \le t \le 7$ hours, the graph consists of two line segments. The slope of the line on the interval $5 \le t \le 6$ is $-3\frac{\text{miles}}{\text{hour}}$ and the slope of the line on the interval $6 \le t \le 7$ is $-2\frac{\text{miles}}{\text{hour}}$.

When a graph is given on a grid, points at the intersection of two grid lines can be read from the graph. If the grid is not provided, only labeled points should be read from the graph. All other points should be determined using analytical methods.

The answers to this lesson are restated in function notation so that the activity can be extended for upper level courses.

Suggested modifications for additional scaffolding include the following:

Table	Fill in the distance in miles from home for one or more rows of the table
8	On the graph, outline the smaller of the two similar triangles used to determine the later time when Ann is 4 miles from home. Monitor and assist as the student draws the larger triangle and sets up the proportion needed to answer this question.
11	Instruct the student to calculate each unit rate and record it on the graph. Supply the rates for the first two segments.

NMSI CONTENT PROGRESSION CHART

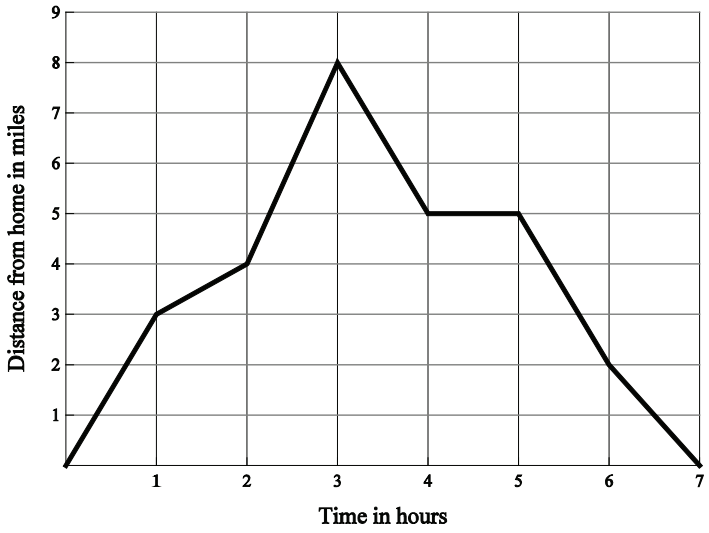
In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	For a continuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.
Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios and/or equations to determine a value on a continuous piecewise function.	Use ratios and/or equations to determine a value on a continuous or discontinuous piecewise function.	Use ratios and/or equations to determine a value on a continuous or discontinuous piecewise function.

Interpreting Distance Graphs

Ann went for a walk on Saturday. When the timing started, she was already traveling at the given rate. She walked in a straight line away from home then she returned home along the same path. On her way home, she stopped for lunch. The graph shows her distance from home at any given time during the walk. Complete the table, and then use the graph and table to answer the following questions.



Time, *t*, in hours	Distance, *d*, from home in miles
0	
1	
2	
3	
4	
5	
6	
7	

1. At what time is Ann's distance from home 0 miles? Give a reason that support your answer.

2. What is Ann's maximum distance from home? At what time did she turn around to return home? Give reasons that support your answers.

3. When does Ann stop for lunch? How far from home is she when she stops for lunch? How long does she stop for lunch? Give reasons that support your answers.

4. How many hours is Ann's trip? Give a reason that support your answer.

5. When does Ann walk the fastest? What is her speed in miles per hour? Give reasons that support your answers.

6. When is Ann walking the slowest? What is her speed in miles per hour? Give reasons that support your answers.

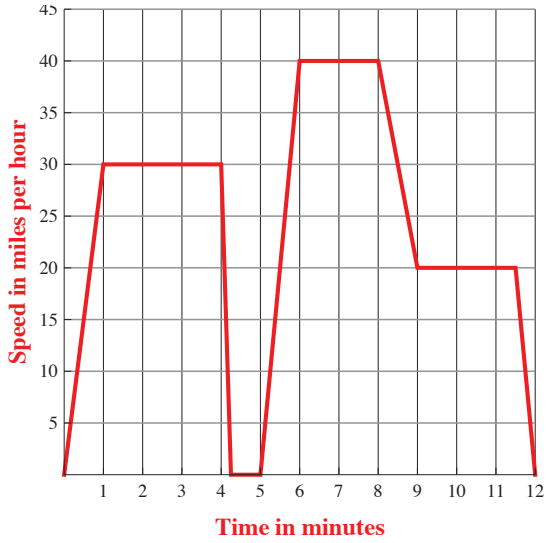
7. What is Ann's distance from home at 6 hours? Give a reason that support your answer.

8. At what times is Ann's distance from home 4 miles? Show the work that leads to your answer.

9. During what times is Ann's distance from home increasing? Give a reason that support your answer.

10. During what times is Ann's distance from home decreasing? Give a reason that support your answer.

11. When is Ann walking at the same rate away from home and toward home? Give a reason that support your answer.



LEVEL

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MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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V – Verbal
A – Analytical
N – Numerical
G – Graphical

Interpreting Rate Graphs

ABOUT THIS LESSON

This lesson provides students an opportunity to use multiple modalities as they investigate a graph that shows speed as a function of time. In addition to answering questions about speed that relate directly to the graph, students analyze information from the speed graph to answer questions about distance traveled. In the context of this lesson, students must carefully use dimensional analysis as they work with speed in miles per hour as it relates to time in minutes.

This lesson focuses on understanding and interpreting how a rate is expressed on a graph. The lesson reinforces proportional reasoning and dimensional analysis. It also enhances student understanding of these standards by developing coherence and connections among these concepts, skills, and mathematical practices.

OBJECTIVES

Students will

- walk a path along a straight line following specific speed and time directions.
- create a graph of the path of the walk and compare it to a speed-time graph of the walk.
- analyze a speed-time graph for specific characteristics.
- determine distance traveled during a time interval with constant speed.
- apply unit rates to determine the coordinates of a point not at a grid intersection.

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill.

Targeted Standards

6.EE.9: Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation to represent the relationship between distance and time.*
See questions 1-6, 9-11

7.RP.2b: Recognize and represent proportional relationships between quantities.
(b) Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
See questions 6-7, 11

Reinforced/Applied Standards

6.RP.3d: Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
(d) Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.
See questions 12-13

5.G.2: Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.
See question 8

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.6: Attend to precision.
Students use accurate units and correct mathematical language in their analysis of the graph.

In questions 12 and 13, students apply dimensional analysis to calculate the distance traveled.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Determine the coordinates of a point in Quadrant I
- Interpret the meaning of a point on a coordinate plane in the context of a situation
- Use dimensional analysis in real-world situations

ASSESSMENTS

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – 6th Grade Free Response Questions
- Analysis of Functions: Piecewise Graphs – 6th Grade Multiple Choice Questions
- Analysis of Functions: Piecewise Graphs – 7th Grade Free Response Questions
- Analysis of Functions: Piecewise Graphs – 7th Grade Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages

TEACHING SUGGESTIONS

Before beginning the student activity pages, ask a student to walk on a horizontal line across the front of the room as you read the following scenario involving speed. When the walk is complete, ask the class to draw a picture of the path. The speed information models part of the speed-time graph shown in the lesson.

- Start walking away from your starting point, increasing your speed at a constant rate.
- Walk at a constant rate.
- Continue walking, decreasing your speed at a constant rate.
- Stop to rest.

The picture of the path should be a horizontal line segment of undetermined length.

The graph in the lesson shows speed as a function of time. The independent variable measures time and the dependent variable measures speed. This graph does not represent Ann's path.

Questions 1 – 8 ask directly about speed. Since speed is the label on the vertical axis, these questions can be answered by reading points directly from the graph. At the middle grades level when a graph is given on a grid, points at the intersection of two grid lines can be read from the graph. If the grid is not provided, only labeled points should be read from the graph and all other points should be determined using analytical methods. Questions 9 – 11 ask about distance, but the vertical axis of the graph is labeled speed; therefore, additional thinking and/or calculation is required in order to answer these questions. Likewise questions 12 and 13 ask how far Ann travels in specific numbers of minutes. Speed is given in miles per hour, so these questions are most efficiently answered using dimensional analysis.

For all questions, remind students that a complete answer includes the appropriate units with any numerical value and a statement of the student's reasoning.

Suggested modifications for additional scaffolding include the following:

1 Modify the question to provide a graphical clue, such as "During what times does the graph have the greatest speed values?"

7 On the graph, outline the two similar triangles used to determine the first time when Ann's speed is 35 miles per hour. Monitor and assist as the student sets up the proportion needed to answer this question. Continue to monitor and assist as the student identifies the similar triangles used to determine the second time when Ann's speed is 35 miles per hour.

12 Provide a dimensional analysis template, similar to the answer key with the units given and blanks for each of the numerical values, to help the student determine a strategy and organize the work.

NMSI CONTENT PROGRESSION CHART

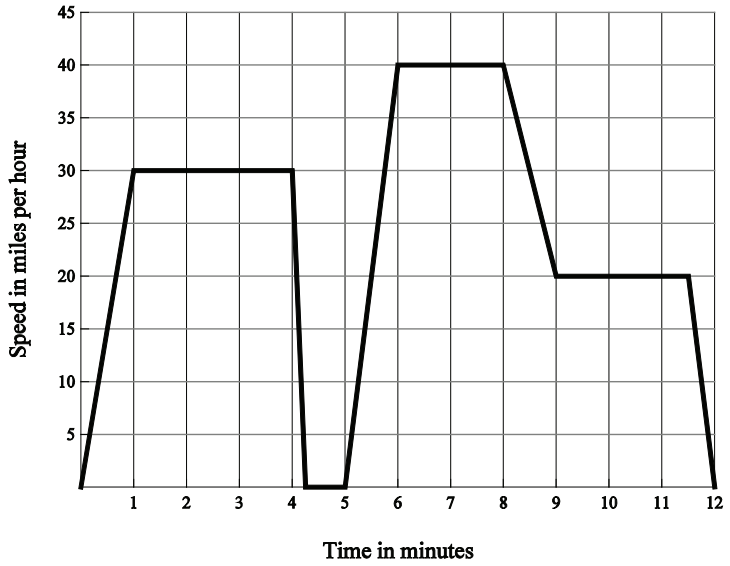
In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	For a continuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.
Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios and/or equations to determine a value on a continuous piecewise function.	Use ratios and/or equations to determine a value on a continuous or discontinuous piecewise function.	Use ratios and/or equations to determine a value on a continuous or discontinuous piecewise function.

Interpreting Rate Graphs

Ann drove in a straight line to school on Monday. A graph of her speed in miles per hour as she traveled away from home is shown. Use the graph to complete the table of values.



Time in minutes	Speed in miles per hour
0	
1	
4	
4.25	0
5	
6	
8	
9	
11.5	20
12	

1. During what times is Ann driving the fastest? Give a reason to support your answer.

2. During what times is Ann stopped? Give a reason to support your answer.

3. During what times is Ann driving at a constant speed? Give a reason to support your answer.

4. When is Ann's speed decreasing? Give a reason to support your answer.

5. When is Ann's speed increasing? Give a reason to support your answer.

6. When is Ann's speed increasing the fastest? Give a reason to support your answer.

7. At what time is Ann driving 35 mph? Show the work that leads to your answer.

8. What is Ann's speed at 7 minutes? Give a reason to support your answer.

9. When is Ann's distance from home increasing? Give a reason to support your answer.

10. When is Ann's distance from home staying the same? Give a reason to support your answer.

11. When is Ann's distance from home increasing the fastest? Give a reason to support your answer.

12. How far does Ann travel between 1 and 4 minutes? Show work to support your answer.

13. How far does Ann travel between 6 and 8 minutes? Show work to support your answer.

All Aboard!

ABOUT THIS LESSON

This lesson presents students with two distance-time graphs and asks them to read and interpret various characteristics of the graph in the context of a real-world scenario. By relating the graph and creating a table for each train over different time intervals, students can draw conclusions about particular data points, time and distance intervals, and even speed. This lesson focuses on reading, interpreting, and understanding a graph of distance with respect to time. The lesson reinforces using units in both the work and the answer of any given question and giving a specific reason to support an answer.

LEVEL

Grade 6 or 7 in a unit on coordinate graphing or rate of change

MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

OBJECTIVES

Students will

- analyze a distance-time graph for specific characteristics.
- use unit rates to determine the speed of the train for specific time intervals.
- determine total distance traveled.
- interpret a distance-time graph and relate it to the path traveled.
- create and interpret tables based on two distance-time graphs.

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill.

Targeted Standards

6.RP.3d Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
(d) Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.
See questions 1, 7-12

7.RP.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas, and other quantities measured in like or different units.
For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction 1/2/1/4 miles per hour; equivalently 2 miles per hour.
See questions 1, 7-12

7.RP.2b Recognize and represent proportional relationships between quantities.
(b) Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
See questions 1, 7-12

Reinforced/Applied Standards

6.NS.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.
See questions 1-12

7.NS.3 Solve real-world and mathematical problems involving the four operations with rational numbers. *(Footnote: Computations with rational numbers extend the rules for manipulating fractions to complex fractions.)*
See questions 1-12

5.G.2 Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.
See questions 1-13

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.1 Make sense of the problem and persevere in solving them.
Students must apply all given constraints outlined in the situation and must understand the difference between clock time and travel time in order to determine the speed of the train for each time interval.

MP.2 Reason abstractly and quantitatively.
Students interpret key features of a distance-time graph in the context of the situation.

MP.6 Attend to precision.
Students use correct mathematical language and accurate units in their analysis of the graph.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Dimensional analysis in real-world situations
- Interpret a point in Quadrant 1 in terms of the units on the graph

ASSESSMENTS

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – 6th Grade Free Response Questions
- Analysis of Functions: Piecewise Graphs – 6th Grade Multiple Choice Questions
- Analysis of Functions: Piecewise Graphs – 7th Grade Free Response Questions
- Analysis of Functions: Piecewise Graphs – 7th Grade Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages

ACKNOWLEDGEMENTS

Mishra, Alo. "Bullet train." *Wikimedia Commons* Image 03 30 2010. Web, https://commons.wikimedia.org/wiki/File:Bullet_train.jpg.

TEACHING SUGGESTIONS

This lesson provides an opportunity to build or reinforce the idea that the distance-time graph is not the path being traveled, but instead, the distance an object is from its initial starting position at each particular time. Students are provided with a graph of two different train schedules, the Commuter Train and the Express Train, along with the distance the train is from the Maple Avenue Train Station for the morning commute.

Before students begin question 1, ask students to analyze different intervals on the graph using precise mathematical language and vocabulary. For example, ask students to explain the meaning in context for which the graph is increasing over a given time interval. Repeat for decreasing and constant. It may also be necessary to review the difference between a time interval and the number of minutes the train is traveling over a time interval. Students may find it helpful to indicate the names of the stations by the appropriate miles on the vertical axis to help answer specific questions in the lesson.

Once students have completed both tables of values, they will utilize the information in the tables to answer the remaining questions and make comparisons. Remind students that a complete answer includes the appropriate units with any numerical value and a statement of the student's reasoning.

Suggested modifications for additional scaffolding include the following:

1 Provide a dimensional analysis template for calculating the speed the train is traveling between stations, with the units given and blanks for each of the numerical values, to help the student determine a strategy and organize the student work.

Fill in the time interval columns for one or more rows of each table.

6 Provide a fill-in-the-blank template such as "From 9:20 to 9:45 and from 10:15 to 10:30, the commuter train is traveling at 6.8 miles per hour. From _____ to _____ and from _____ to _____ the commuter train is traveling at 3.4 miles per hour."

8 Fill in one or more rows of the table.

NMSI CONTENT PROGRESSION CHART

In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
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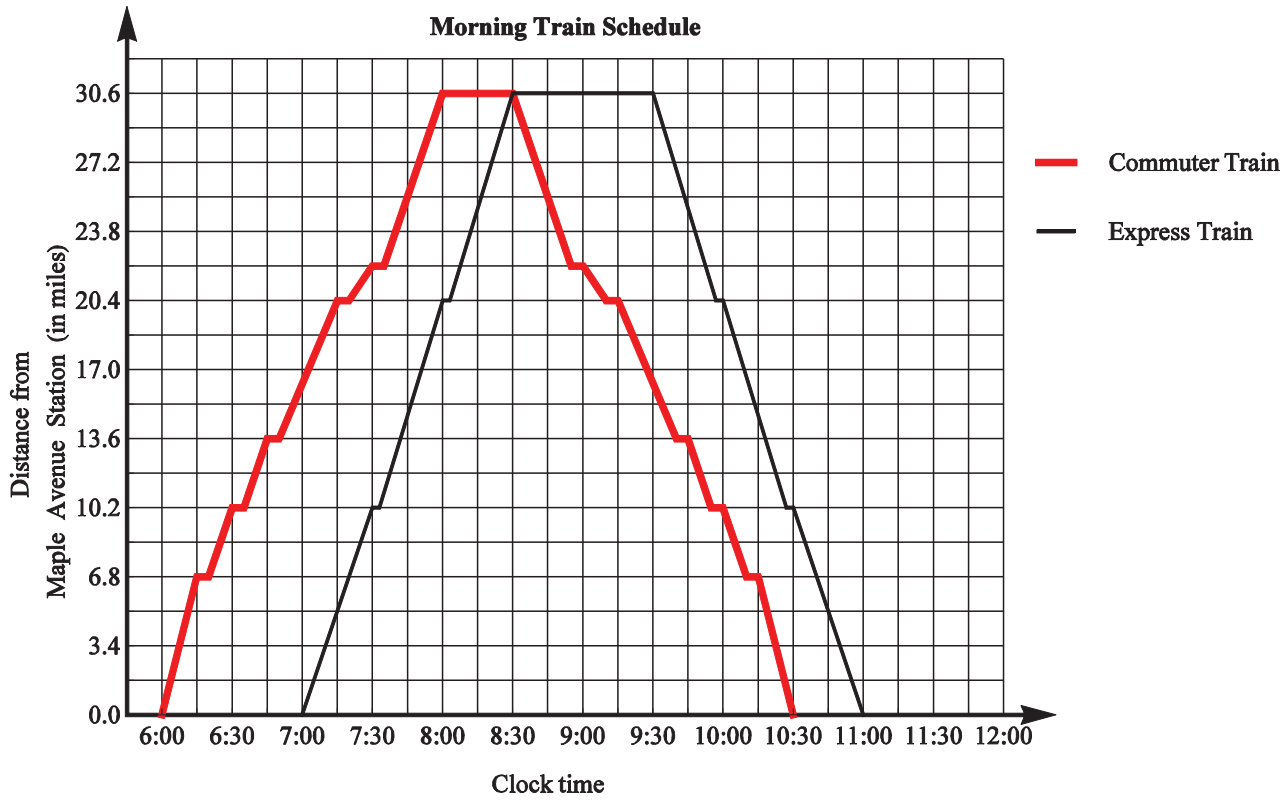
Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios to determine a value that is not at a grid (lattice) point on a piecewise graph involving a real-world application (including rate graphs).	Use ratios and/or equations to determine a value on a continuous piecewise function.	Use ratios and/or equations to determine a value on a continuous or discontinuous piecewise function.	Use ratios and/or equations to determine a value on a continuous or discontinuous piecewise function.

All Aboard!

The Quick Rail Train Company has added a new train route between the Maple Avenue Station and the Bradford Avenue Station. The Commuter Train leaves the Maple Avenue Station at 6:00 a.m. and the Express Train leaves at 7:00 a.m. The map represents how far the train is at any given time from the Maple Avenue Train Station. The commuter train stops at every station for 5 minutes for passenger loading and unloading. The Express Train does not stop at the Ash Lane, Pine Street, or Spruce Lane Stations, but stops for 2 minutes at the Plum Street Station and at the Pecan Street Station. Each train stops at the Bradford Avenue Station for maintenance, cleaning, and a break for the train engineer.



Station Location	Distance from Maple Avenue Station	Commuter Train Stops at Station	Express Train Stops at Station
Ash Lane	6.8 miles	Yes	No
Plum Street	10.2 miles	Yes	Yes
Pine Street	13.6 miles	Yes	No
Pecan Street	20.4 miles	Yes	Yes
Spruce Lane	22.1 miles	Yes	No
Bradford Avenue	30.6 miles	Yes	Yes

1. Complete the tables to show when each train is traveling and when it is stopped, the distance between the stations, and the speed of the train between stations for the trip from Maple Avenue to Bradford Street. Round speed in miles per hour to tenths when necessary.

Commuter Train					
Destination	Time Interval When Train is Traveling	Minutes the Train is Traveling	Distance Between Train Stations in Miles	Average Speed the Train is Traveling Between Stations in Miles per Hour	Time Interval When Train is Stopped
Maple Ave to Ash Lane	6:00 – 6:15	15 minutes	6.8 miles	27.2 $\frac{\text{miles}}{\text{hour}}$	6:15 – 6:20
Ash Lane to Plum Street					
Plum Street to Pine Street					
Pine Street to Pecan Street					
Pecan Street to Spruce Lane					
Spruce Lane to Bradford Street					

Express Train					
Destination	Time Interval When Train is Traveling	Minutes the Train is Traveling	Distance Between Train Stations in Miles per Hour	Average Speed the Train is Traveling Between Stations in Miles per Hour	Time Interval When Train is Stopped
Maple Ave to Plum Street	7:00 – 7:30	30 minutes	10.2 miles	20.4 $\frac{\text{miles}}{\text{hour}}$	7:30 – 7:32
Plum Street to Pecan Street					
Pecan Street to Bradford Street					

2. How many total miles does the Commuter Train travel from 6:00 a.m. to 10:30 a.m.?

3. Riding the Commuter Train, how many total minutes does it take to arrive at the Pecan Street Station once it departs from the Maple Avenue Station?

4. How many total minutes does the Commuter Train stop from the time it leaves the Maple Avenue Station to when it returns at 10:30 a.m.?

5. How many total minutes is the Express Train stopped from 7:00 a.m. until 8:30 a.m.?

6. From 8:30 a.m. to 10:30 a.m., identify the time intervals when the Commuter Train is traveling the same average speed between stops?

7. A passenger boards the train at the Ash Lane Station and exits at the Pecan Street Station. Which train did the passenger board? How many minutes is the passenger on the train from the time it leaves the Ash Lane Station it arrives at the Pecan Street Station? How many miles does the passenger travel?

8. For each time interval, identify whether the distance the train is away from the Maple Avenue Station is increasing, decreasing, or remaining constant.

Train	**Time Interval**	**Increasing, Decreasing, or Constant**
Commuter	6:35 to 6:45	
Express	7:30 to 7:32	
Express	10:00 to 10:28	
Commuter	9:45 to 9:55	
Express	7:45 to 8:00	
Express	9:00 to 9:30	
Commuter	8:30 to 8:55	
Express	8:30 to 9:00	

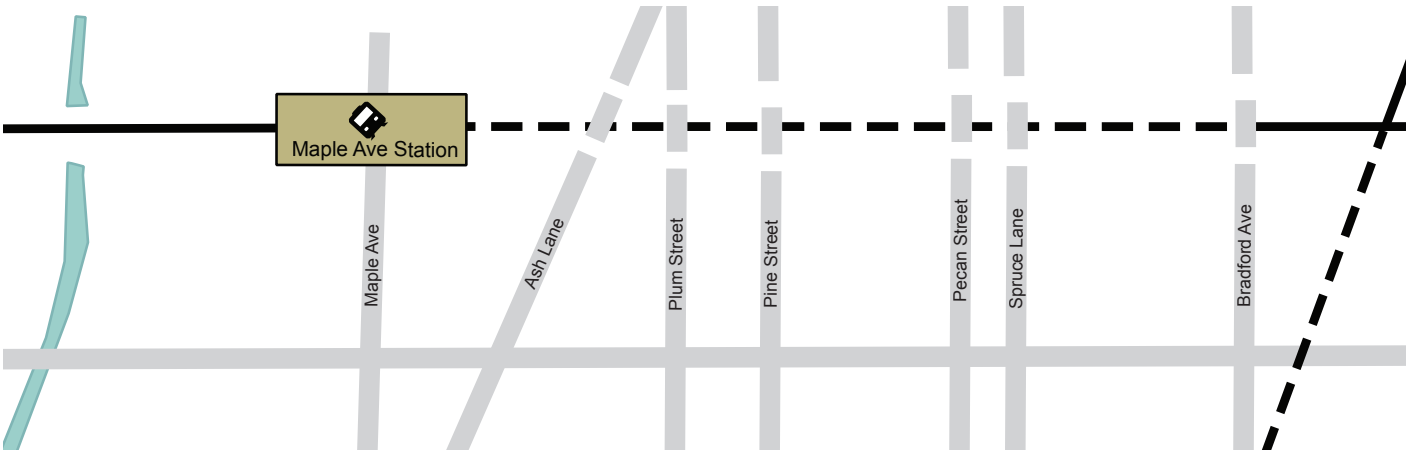
9. Determine the average speed of each train, in miles per hour, from the Maple Avenue Station to the Bradford Avenue Station. Show the work leading to your answer.

10. How fast is the Commuter Train traveling from 9:15 a.m. to 9:40 a.m.?

11. How fast is the Express Train traveling as it passes through the Ash Street Station on its return to the Maple Avenue Station?

12. On Thursday, the Commuter Train was running on schedule until it arrived at the Pecan Street Station on its return to the Maple Avenue Station. The train was delayed until 9:25 a.m. Determine how much faster the train will need to travel in order to make it to Pine Street Station on time.

13. The Quick Rail Train Company wants to post an illustration of the train stops for both trains. Use the city map to create an illustration to show passengers where each train stops along the rail line.



Piecewise Functions

ABOUT THIS LESSON

This lesson introduces students to piecewise functions using a distance-time graph of a real-world scenario. Students are asked to read information from the graph and an accompanying data table and to interpret the meaning in the context of the situation. Students also write a piecewise linear function for the graph and use the piecewise function to determine additional information analytically.

OBJECTIVES

Students will

- analyze and interpret a distance-time graph.
- write a piecewise linear function.
- compute average rate of change.
- create a scenario to match a given distance-time graph.

LEVEL

Grade 8, Algebra 1, or Math 1 in a unit on graphing and writing linear functions

MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

TEACHER PAGES

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (★) at the end of a specific standard indicates that the high school standard is connected to modeling

Targeted Standards *(if used in Grade 8)*

8.F.4: Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (*x, y*) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
See questions 3, 5, 7

Reinforced/Applied Standards *(if used in Grade 8)*

8.F.5: Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.
See questions 3-4, 8-9

Targeted Standards *(if used in Algebra 1)*

F-IF.6: Calculate and interpret the average rate of change of a function over a specified interval.
See questions 3, 7

F-BF.1: Write a function that describes a relationship between two quantitites.★
See questions 5

Reinforced/Applied Standards *(if used in Algebra 1)*

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
See questions 3-4, 8-9

F-IF.5: Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★
See questions 5, 9

A-CED.1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.★
See questions 6

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.6: Attend to precision.
In question 9, students use accurate mathematical language with specific details in creating a context for a given graph.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Interpret a distance graph
- Calculate a rate of change
- Write a linear function

ASSESSMENTS

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – Algebra 1 Free Response Questions
- Analysis of Functions: Piecewise Graphs – Algebra 1 Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Applet for exploring parameters in piecewise functions: http://www.mathdemos.org/mathdemos/piecewise/Examples/Java_continuity.html
- NMSI video clips on creating piecewise functions on the TI-84 and TI-Nspire

TEACHING SUGGESTIONS

This lesson is an excellent introduction to piecewise functions. If it is used in this manner, the teacher may want to complete the lesson with the entire class working together. Before beginning the student activity pages, model the path of the bicycle by having a student walk along a horizontal line across the front of the room while you describe the information from the graph of the bicycle trip. Scale both distance and time to fit into the classroom.

- Walk forward 30 "miles" at a constant rate during a 2 "hour" time period.
- Stop and rest for 2 "hours."
- Walk forward 30 "miles" at a slower constant rate during a three "hour" time period.

When the walk is complete, ask the class to sketch a graph of the path. The path will be a horizontal line. Discuss that the distance-time graph of the bicycle trip is not a replica of the path followed by the bicycles.

The piecewise functions in this lesson are continuous. Note that in the questions and answers, the written equations include the shared endpoints in each of the domain intervals. (For example, see the answer for question 5.) Students may argue that including the shared endpoint in both intervals creates an equation that is not a function; however this is simply not true. Teachers should explain that, since evaluating both function pieces at their common x-value produces the same y-value, writing the equation in this way does not contradict the definition of a function which states for each x-value, there is only one y-value. (The Test Development Committee of the College Board has used this approach on past exams. For an example, see 2006 AB/BC 4 Form B.)

You may wish to support this activity with TI-Nspire™ technology. See *Graphing Piecewise Functions* in the NMSI TI-Nspire Skill Builders.

Suggested modifications for additional scaffolding include the following:

5 Complete two of the four blanks; for instance, fill in the equation and leave the domain blank for the second piece, fill in the domain and leave the equation blank for the third piece.

6 Supply a copy of the graph with the horizontal line, $d = 22$, drawn.

7 Insert a table similar to the one in the prompt with rows for $t = 1$ and $t = 3$. Provide a model for calculating average rate of change, such as $\frac{d(3) - d(1)}{3 - 1} \frac{\text{miles}}{\text{hour}}$.

8 Provide examples of correct and incorrect answer graphs.

NMSI CONTENT PROGRESSION CHART

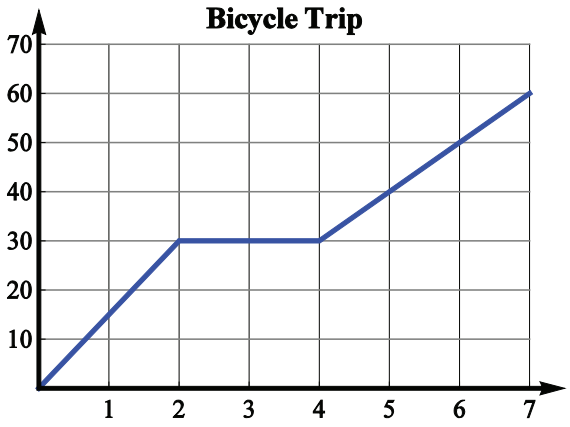
In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	For a continuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.
		Write a piecewise linear function based on a graph.	Write a piecewise function based on a graph (may include portions of a circle and linear pieces).	Write a piecewise function based on a graph (may include portions of a circle and/or linear and parabolic pieces).	Write a piecewise function based on a graph (may include portions of conic sections and/or linear, rational, or trigonometric pieces).

Piecewise Functions

The Free Wheelers Bicycle Club went on a Saturday bicycle trip. The graph shows the relationship between time in hours and distance in miles from the starting point for one female club member. She traveled away from her start point in a straight line. When the timing started, she was already traveling at the given rate. Use the graph of *d* and the table to answer the following questions.



t	*d*
0	0
2	30
4	30
7	60

1. For this situation, _____ is a function of _____.

2. Write these labels on the appropriate axes with the units, where distance is measured in miles and time is measured in hours.

3. During which interval is she moving the fastest? Explain how you selected your answer.

4. Describe what is happening on the bicycle trip for the first 2 hours.

5. The graph consists of three distinct line segments. Write an equation for each part: one for the interval $t = 0$ hours to $t = 2$ hours, one for the interval $t = 2$ hours to $t = 4$ hours, and one for the interval $t = 4$ hours to $t = 7$ hours. This graph is called a piecewise function and would be written as

$$d = \begin{cases} 15t & \text{for } 0 \le t \le 2 \\ \underline{\qquad\qquad} & \text{for } \underline{\qquad\qquad} \\ \underline{\qquad\qquad} & \text{for } \underline{\qquad\qquad} \end{cases}$$

6. When is $d = 22$ miles? Show the work that leads to your conclusion and interpret the meaning of the answer.

7. What is the average speed for the interval $t = 1$ hour to $t = 3$ hours?

8. Suppose that at the end of the fourth hour, the bicyclist decides to go back to her starting position. Illustrate what the graph might look like for the return trip.

9. Create a story for the original graph of the function d for the interval $t = 0$ hours to $t = 7$ hours.

Putting the Pieces Together

ABOUT THIS LESSON

This lesson provides students an opportunity to match a scenario, with the corresponding piecewise graphs for distance-time and speed-time, and then write the equation of the piecewise defined function for each graph. Students are asked to complete a graph based upon a scenario, read information from the graph and an accompanying data table, and interpret the meaning in the context of the situation. Students also write a piecewise linear function for the graph and use the piecewise function to determine additional information analytically.

OBJECTIVES

Students will

- analyze and interpret a distance-time graph.
- write a piecewise linear function.
- compute average rate of change.
- match a scenario, a distance-time graph, and a speed-time graph.

LEVEL

Grade 8 as a part of a unit on graphing and writing linear functions

Algebra 1 or Math 1 as a review of graphing and writing linear functions

MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

TEACHER PAGES

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill.

Targeted Standards (*if used in Grade 8*)

8.F.4 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
See questions 1, 2d-e, 3c-d

Reinforced/Applied Standards (*if used in Grade 8*)

8.F.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.
See questions 1, 2a-c, 3

Targeted Standards (*if used in Algebra 1*)

F-IF.6 Calculate and interpret the average rate of change of a function over a specified interval.
See questions 1, 2d-e, 3c-d, 3j

F-BF.1 Write a function that describes a relationship between two quantities.⋆
See questions 1, 2d

Reinforced/Applied Standards (*if used in Algebra 1*)

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*⋆
See questions 1, 2a

F-IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*⋆
See questions 1, 2d, 3j

A-CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.⋆
See questions 1, 2d-e

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.1 Make sense of the problems and persevere in solving them.
Students analyze the constraints and interpret the context of each situation in order to determine a reasonable starting point and use the information appropriately throughout the problem.

MP.2 Reason abstractly and quantitatively.
Students use their knowledge of rate of change and linear functions to connect a verbal description, the distance-time and speed-time graph, and write the piecewise function for each graph.

MP.3 Construct viable arguments and critique the reasoning of others.
In question 3c, students analyze the graph to prove whether Miguel's conjecture is true or false.

MP.6 Attend to precision.
Students use accurate units and correct mathematical language in their analysis of the graph.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Interpret a distance graph
- Calculate a rate of change
- Write and interpret a linear function

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – Algebra 1 Free Response Questions
- Analysis of Functions: Piecewise Graphs – Algebra 1 Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Stapler
- Tape
- Ruler
- Applet for exploring parameters in piecewise functions: http://www.mathdemos.org/mathdemos/piecewise/Examples/Java_continuity.html
- NMSI video clips on creating piecewise functions on the TI-84 and TI-Nspire

TEACHING SUGGESTIONS

This lesson can be used in Grade 8, Algebra 1, or Math 1 as part of a unit on linear functions since there is a close relationship between 8.F.5 and F-IF.4, 8.F.4 and F-IF.6, and 8.F.4 and F-BF.1. The Grade 8 standards introduce analyzing functions and calculating average rates of change, and then the high school standards build on this foundational understanding.

As students begin the activity, discuss different approaches for the matching activity. Some students may choose to match all the piecewise graphs with the descriptions, while others may complete the entire row before moving to the next description. Take time to discuss the units for the piecewise graphs as well as the relationship between the description, the graph, and the function. It is also important for students to identify a connection between the distance-time graph and the speed-time graph. Since speed is the absolute value of velocity, students should infer that the slope of each interval from the distance-time graph can be used to determine speed for each corresponding interval. Remind students to fill in the blanks in the description column.

In question 2, students must complete the distance-time graph based on the scenario. It is important to discuss with students that they are completing a distance-time graph and not a graph of the path Krystal walks. In 2e, Krystal returns home at a constant rate given in kilometers per hour. Since the original scenario is given in meters per minute, this question is most efficiently answered using dimensional analysis.

In question 3, students relate the paths of two runners to their corresponding distance-time graphs. If students are not familiar with the layout of the yard markers on a football field, discuss that, if a runner begins from the south goal line and reaches the 30 yard marker closest to the north goal line, then the runner is on the 30 yard line but has traveled 70 yards. For each time interval, students determine the rate for each runner and then utilize the graph, the path, and the table to answer the remaining questions. In part 3i, students complete the distance-time graph for each runner. It is important to point out that the runners rest for 5 seconds before they return to the south goal line.

For all questions, remind students that a complete answer includes the appropriate units with any numerical value and a statement of the student's reasoning.

You may wish to support this activity with TI-Nspire™ technology. See *Graphing Piecewise Functions* in the NMSI TI-Nspire Skill Builders.

Suggested modifications for additional scaffolding include the following:

1 Provide a list of possible equations to select from for the piecewise functions.

2d Complete part of the piecewise function; for instance, fill in the equation and leave the domain blank for the second piece, fill in the domain and leave the equation blank for the third piece.

2e Provide a dimensional analysis template, similar to the answer key with the units given and the blanks for each of the numerical values, to help the student determine a strategy and organize the work.

3a Illustrate the first two intervals on the field for both runners.

NMSI CONTENT PROGRESSION CHART

In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

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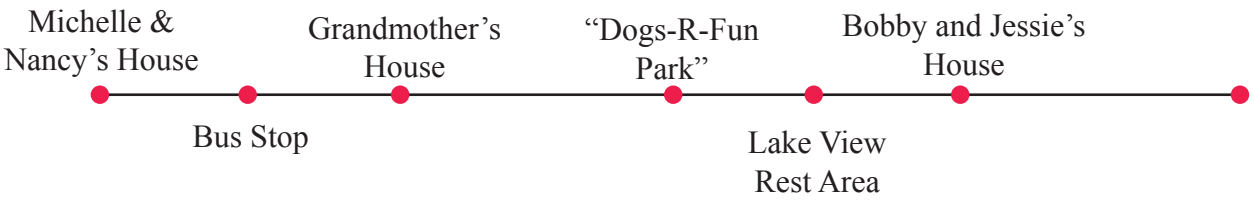
		Write a piecewise linear function based on a graph.	Write a piecewise function based on a graph (may include portions of a circle and linear pieces).	Write a piecewise function based on a graph (may include portions of a circle and/or linear and parabolic pieces).	Write a piecewise function based on a graph (may include portions of conic sections and/or linear, rational, or trigonometric pieces).

Putting the Pieces Together

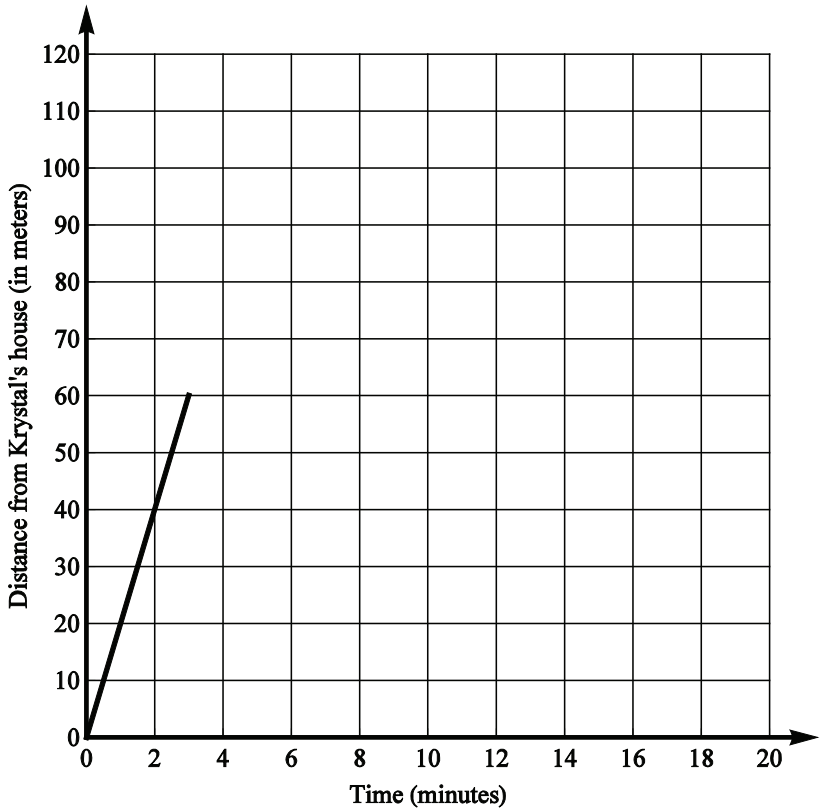
1. Michelle and Nancy's parents have told them that they must exercise each day after school. They have allotted 1 hour each day for their workouts. This week they decided to describe their workout, create a graph to map their routine and to track their speed, and then write a set of equations for each graph. They like to run, walk, or bike each day in their neighborhood. From Michelle and Nancy's house, the Dogs-R-Fun park is 2 miles away, Uncle Bobby and Aunt Jessie's house is 3 miles away, and Grandmother lives 1 mile away. To ride the bus to school in the morning, they have to walk $\frac{1}{2}$ a mile to the Bus Stop. The Lake View Rest Area is exactly half way between the Dogs-R-Fun Park and Uncle Bobby and Aunt Jessie's House.

   Based upon the verbal description of their exercise for the day, determine the distance-time graph that matches the description and tape it in the appropriate blank and then complete the piecewise-defined function for the distance-time graph. Match the graph of the speed-time based on the distance-time graph, and complete the piecewise function for the speed graph. Fill in the blanks for the description as you work through the activity.



Description	Piecewise Graph	Piecewise Function for Distance/Time Graph	Speed Graph	Piecewise Function for Speed Graph
On Monday, Michelle and Nancy left their house, ran to the dog park at a constant rate of ____ miles per hour, stopped for 10 minutes and then ran to Bobby and Jessie's house for dinner at a rate of ___ miles per hour.		$d(t)=\begin{cases} \_\_\_\_\_, & 0\le t\le \frac{1}{2} \\ \_\_\_\_\_, & \frac{1}{2}\le t\le \frac{2}{3} \\ 3(t-1)+3, & \frac{2}{3}\le t\le 1 \end{cases}$		$s(t)=\begin{cases} \_\_\_\_\_, & 0<t<\frac{1}{2} \\ \_\_\_\_\_, & \frac{1}{2}<t<\frac{2}{3} \\ 3, & \frac{2}{3}<t<1 \end{cases}$
On Tuesday, Nancy and Michelle were at Grandmother's house when they began their exercise. They left Grandmother's house and first walked home at a constant rate of _____ miles per hour. When they got home, Mom told them to go walk the dog. They were home for 10 minutes before they left for the dog park. They walked to the park at a constant rate of 6 miles per hour.		$d(t)=\begin{cases} \_\_\_\_\_, & 0\le t\le \frac{1}{2} \\ 0, & \frac{1}{2}\le t\le \frac{2}{3} \\ \_\_\_\_\_, & \frac{2}{3}\le t\le 1 \end{cases}$		$s(t)=\begin{cases} 2, & 0<t<\frac{1}{2} \\ \_\_\_\_\_, & \frac{1}{2}<t<\frac{2}{3} \\ \_\_\_\_\_, & \frac{2}{3}<t<1 \end{cases}$
On Wednesday, Michelle and Nancy left their house, rode their bikes at a constant rate of _____ miles per hour until they arrived at the Lake View Rest Area. They rested for 10 minutes and then rode back home at a constant rate of _____ miles per hour.		$d(t)=\begin{cases} 5t, & 0\le t\le \frac{1}{2} \\ 2.5, & \frac{1}{2}\le t\le \frac{2}{3} \\ \_\_\_\_\_, & \frac{2}{3}\le t\le 1 \end{cases}$		$s(t)=\begin{cases} 5, & 0<t<\frac{1}{2} \\ \_\_\_\_\_, & \frac{1}{2}<t<\frac{2}{3} \\ \_\_\_\_\_, & \frac{2}{3}<t<1 \end{cases}$

Description	Piecewise Graph	Piecewise Function for Distance/Time Graph	Speed Graph	Piecewise Function for Speed Graph
On Thursday, Michelle and Nancy had to take Duke, the dog, for a walk. They left home and walked at a constant rate of ___ miles per hour, stopped for 10 minutes at the dog park to let Duke play, then they returned home at a constant rate of ___ miles per hour.		$d(t)=\begin{cases} 4t, & 0\le t\le \frac{1}{2} \\ 2, & \frac{1}{2}\le t\le \frac{2}{3} \\ \_\_\_\_\_, & \frac{2}{3}\le t\le 1 \end{cases}$		$s(t)=\begin{cases} 4, & 0< t< \frac{1}{2} \\ \_\_\_\_\_, & \frac{1}{2}< t< \frac{2}{3} \\ \_\_\_\_\_, & \frac{2}{3}< t< 1 \end{cases}$
On Friday, Michelle and Nancy decided to take Grandmother's dog for a walk. They left Grandmother's house, walked to the dog park at a constant rate of ____ miles per hour, stopped for 10 minutes at the dog park, and then returned to their home at a constant rate of ____ miles per hour.		$d(t)=\begin{cases} 2t+1, & 0\le t\le \frac{1}{2} \\ 2, & \frac{1}{2}\le t\le \frac{2}{3} \\ \_\_\_\_\_, & \frac{2}{3}\le t\le 1 \end{cases}$		$s(t)=\begin{cases} 2, & 0< t< \frac{1}{2} \\ \_\_\_\_\_, & \frac{1}{2}< t< \frac{2}{3} \\ \_\_\_\_\_, & \frac{2}{3}< t< 1 \end{cases}$



2. The grocery store is located on a straight path 120 meters north of Krystal's house. The following is a description of Krystal's path.
   - She began walking to the store at 9:37 a.m.
   - She walked 60 meters toward the store in 3 minutes
   - She realized she left her wallet on her desk at home
   - She turned around and walked back home at the same speed
   - She spent 1 minute looking for her wallet before finding it on the kitchen counter
   - She biked all the way to the store at twice her original speed

   a. Complete the distance-time graph that accurately represents Krystal's trip to the store.

   b. Determine how many total meters Krystal traveled before reaching the store.

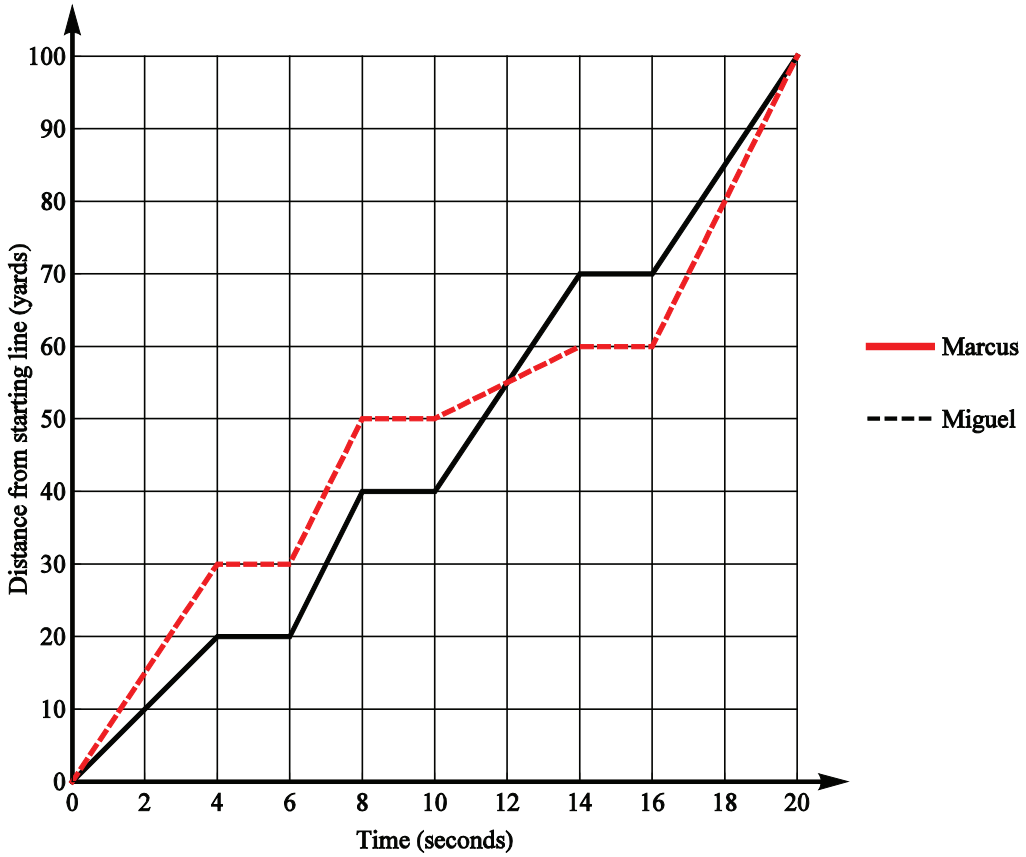
   c. At what time did she arrived at the store?

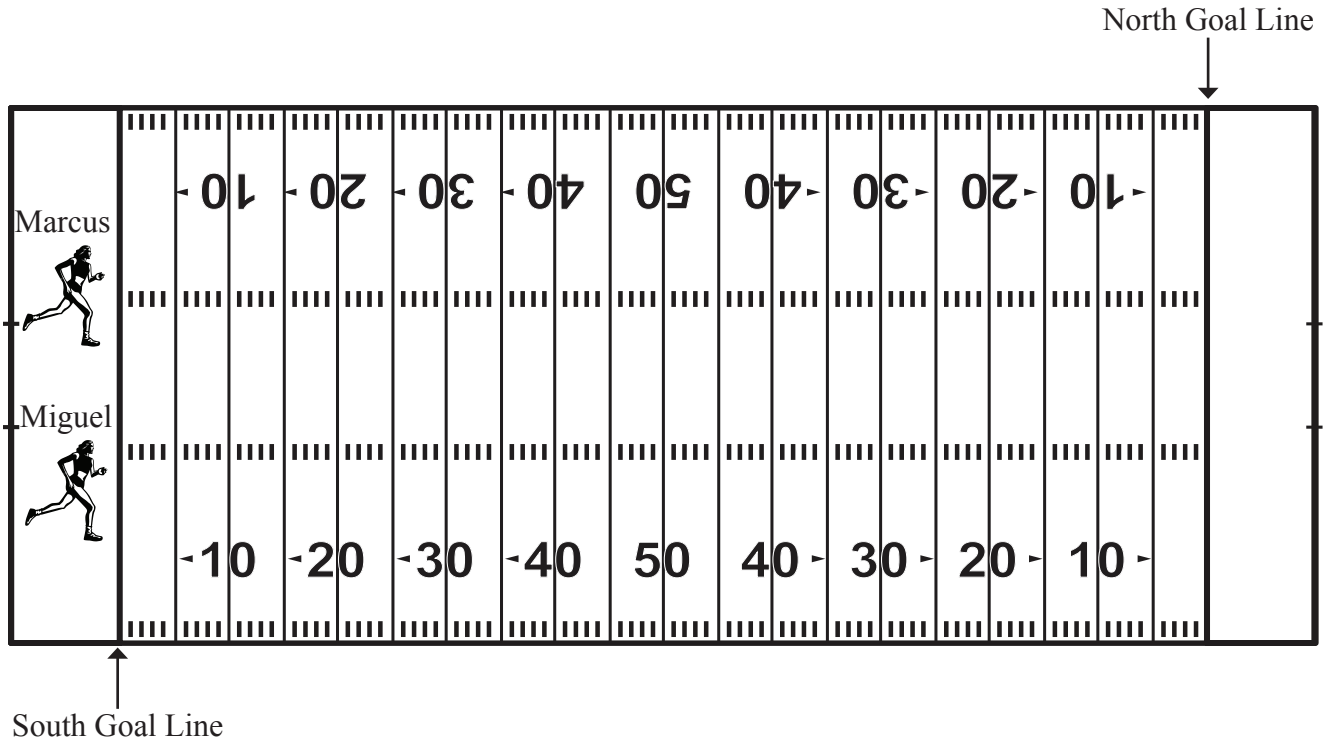
d. Write a piecewise function for the distance time graph.

$$d(t)=\begin{cases} \underline{\qquad\qquad}, & 0 \le t \le 3 \\ \underline{\quad}(t-3)+\underline{\quad}, & \underline{\quad} \le t \le \underline{\quad} \\ \underline{\qquad\qquad}, & \underline{\quad} \le t \le \underline{\quad} \\ \underline{\qquad\qquad}, & \underline{\quad} \le t \le \underline{\quad} \end{cases}$$

e. It takes Krystal 33 minutes to collect her groceries and check-out. She then returns home at a constant rate of 1.2 kilometers per hour. What time does Krystal arrive back home? Show your work that leads to your answer.

3. Marcus and Miguel are participating in after school conditioning for football. Coach Morris has the boys line up on the south goal line of the football field. As they run to the north goal line, they must stop the instant the coach blows the whistle, do a push-up, and then resume running toward the goal line at the next whistle. They must continue the process until they reach the north goal line.





a. Illustrate, on the field, the path of each athlete using an X to notate when the coach blows the whistle and arrows to show the direction the runner is traveling.

b. How far does each runner travel in the first 4 seconds?

c. Miguel looked at the graph and made a conjecture that from $0 \le t \le 4$ and $6 \le t \le 8$, he ran faster than Marcus. Determine if Miguel's conjecture is true. Explain your reasoning.

d. Complete the table to show how fast Marcus and Miguel are running for the indicated time intervals.

Time Interval	Marcus	Miguel
$0 \le t \le 4$		
$4 \le t \le 6$	$0 \frac{\text{yards}}{\text{second}}$	
$6 \le t \le 8$		
$8 \le t \le 10$		
$10 \le t \le 14$	$7.5 \frac{\text{yards}}{\text{second}}$	$2.5 \frac{\text{yards}}{\text{second}}$
$14 \le t \le 16$		
$16 \le t \le 20$		

e. At which yard line are the runners the same distance from the south goal line at the same time? How many yards from the south goal line is each boy at this time?

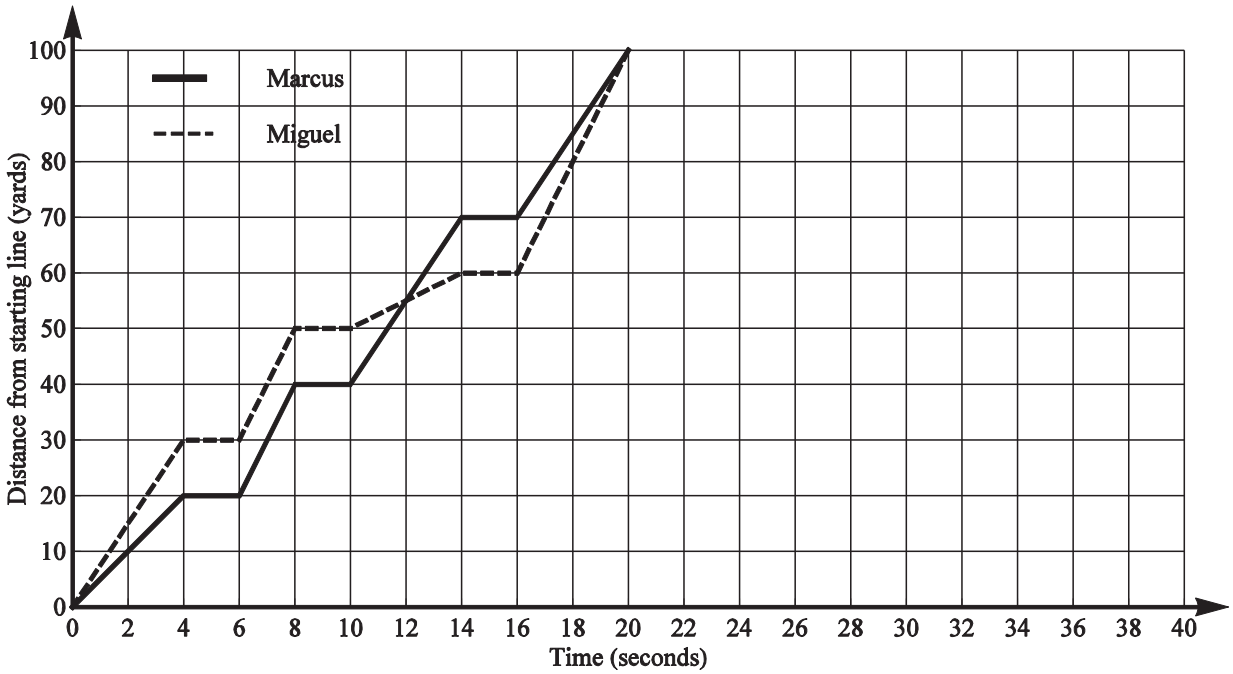
f. Which runner is the farthest from the starting line at $t = 14$ seconds? How many yards is the runner from the south goal line? At which yard line is the runner at this time?

g. At ____ $\le t \le$ _____ Marcus was 30 yards from reaching the north goal line. On this same time interval, Miguel was _____ yards from the north goal line.

h. From the south goal line to the north goal line, the boys spend _____% of their time running, and _____% of their time was spent doing push-ups.

i. After the boys reached the north goal line, the coach gave them 5 seconds to rest and then they had to sprint back to the 50 yard line, do 5 push-ups and then return to the beginning goal line. Continue the graph and the illustration on the football field for each student using the criteria for Marcus and Miguel.

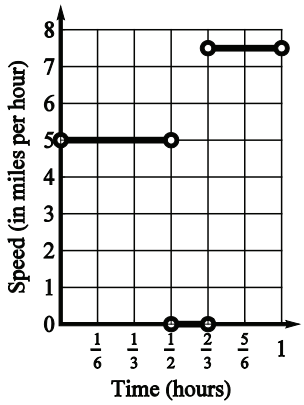
Marcus:	Miguel:
• runs at a constant rate for 4 seconds and then begins his push-ups at the 50 yard line. • finishes his push-ups in 4 seconds. • runs the remaining 50 yards traveling at a rate of 10 yards per second.	• runs to the 50 yard line at a rate of 2.5 yards per second slower than Marcus. • finishes his push-ups in 4 seconds. • finished 2 seconds after Marcus.

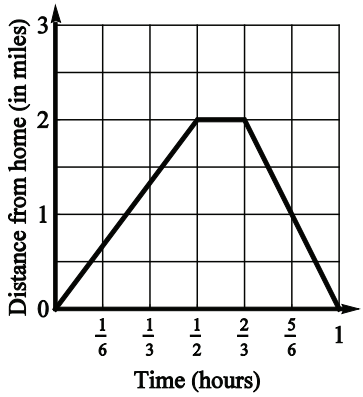


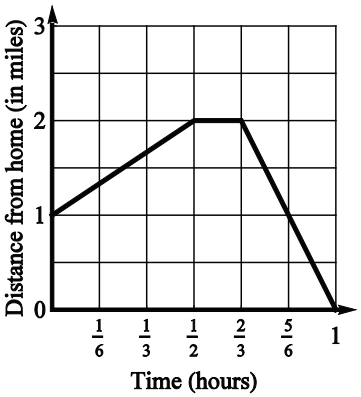
j. Shade in the True or False column to identify whether each statement is valid based on the graph.

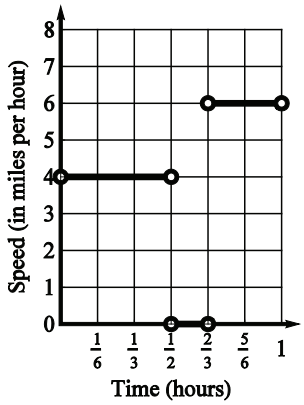
Based on the graph:	**True**	**False**
Marcus always runs faster than Miguel.		
For $0 \le t \le 20$, the average rate of change for both Marcus and Miguel is 5 yards per second.		
The fastest speed that either boy ran was 10 yards per second.		
Miguel and Marcus ran the same total distance of 200 yards.		

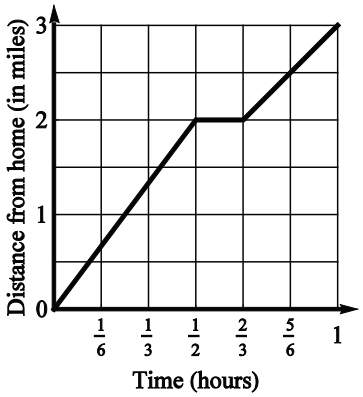
Graphs for Question 1.

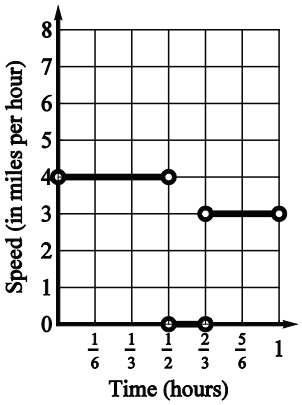


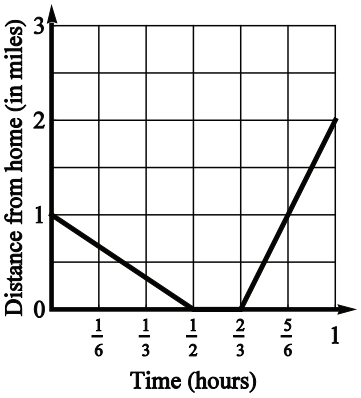


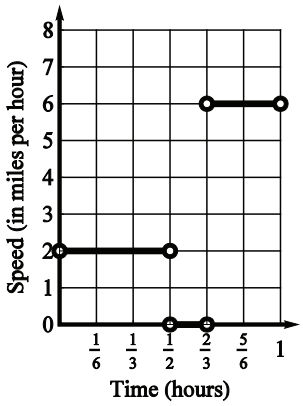


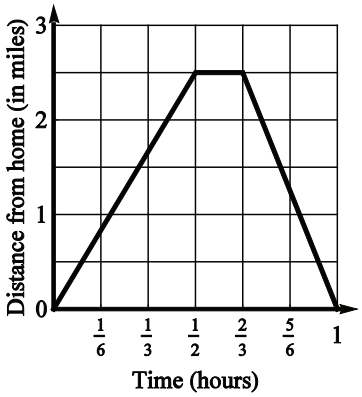


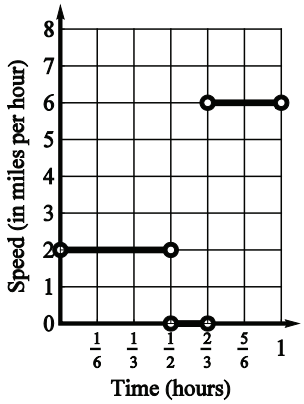


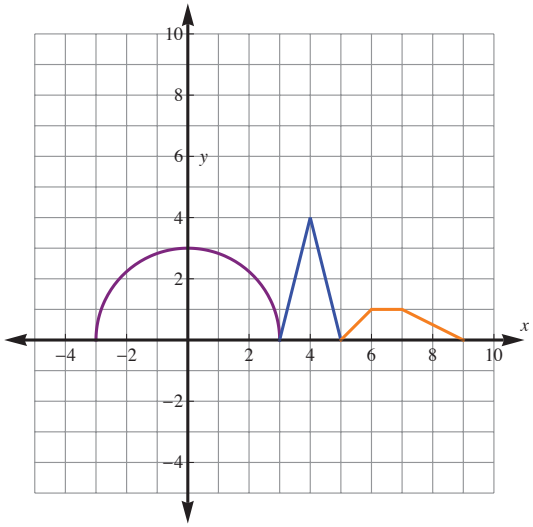












LEVEL

Geometry or Math 2 in a unit on area or transformations

Algebra 2 or Math 3 in a unit on analyzing functions and applying transformations

MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

Analyzing Piecewise Functions

ABOUT THIS LESSON

This lesson encourages students to connect their previously-learned knowledge of piecewise functions, area, and transformations. Working from a graph consisting of semi-circular and linear pieces, with an accompanying table of values, students consider lines of symmetry, analyze the effect of transformations on enclosed area, identify characteristics of the graph, write a piecewise function for the graph, and determine slopes at specific points.

OBJECTIVES

Students will

- analyze attributes of a piecewise function.
- identify lines of symmetry.
- determine the area of a region bounded by a piecewise function and the x-axis.
- apply transformations to graphs of piecewise functions.
- write the equation for a piecewise function.
- apply the understanding that a tangent line is perpendicular to the radius of a circle at the point of tangency to calculating the slope of a line at a point on a circle.

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (★) at the end of a specific standard indicates that the high school standard is connected to modeling

Targeted Standards

G-GPE.7: Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula. ★
See questions 2, 4-6, 8-13

G-CO.2: Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
See questions 6, 9-11, 13, 18

F-BF.3: Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
See questions 6, 9-11, 13, 18

Reinforced/Applied Standards

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*
See questions 14-18

F-BF.1: Write a function that describes a relationship between two quantities. ★
See question 19

A-CED.1: Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* ★
See question 20

F-IF.2: Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
See question 21

F-IF.6: Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★
See question 21

G-C.2: Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.* ★
See question 21

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.7: Look for and make use of structure.
In question 13, students recognize that the additional area created by translating the region up one unit is the same as the area below the x-axis in question 12.

MP.8: Look for and express regularity in repeated reasoning.
In questions 9-11, students generalize the effect that applying a vertical scale factor to a function has on the area bounded by the function.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Calculate areas of circles, triangles, and trapezoids
- Recognize and apply basic transformations
- Use function notation

ASSESSMENTS

The following formative assessment is embedded in this lesson:

- Students engage in independent practice.

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – Geometry Free Response Questions
- Analysis of Functions: Piecewise Graphs – Geometry Multiple Choice Questions
- Analysis of Functions: Piecewise Graphs – Algebra 2 Free Response Questions
- Analysis of Functions: Piecewise Graphs – Algebra 2 Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Patty paper (optional)
- Applet for exploring parameters in piecewise functions: http://www.mathdemos.org/mathdemos/piecewise/Examples/Java_continuity.html
- NMSI video clips on creating piecewise functions on the TI-84 and TI-Nspire
- TI-Nspire companion document "Analyzing_Piecewise_Functions.tns" available for download on the NMSI website.

TEACHING SUGGESTIONS

This lesson is an excellent opportunity for students to apply previously learned knowledge and skills to a new situation. The use of a piecewise function allows for review of various function types within a single activity. Teachers may choose to separate the activity into sections, as appropriate. Questions 1 – 5 address some of the geometric properties of the given graph. Questions 6 – 13 consider a variety of transformations of the original graph and the effect those transformations have on the enclosed area. Questions 14 – 18 focus on identifying key characteristics of the graph, such as increasing/decreasing intervals and absolute maximum/minimum function values. Questions 19 – 21 involve writing equations for each of the pieces of the graph, solving those equations for a particular *y*-value, and determining the slope of the graph at specific points.

When students are asked to determine the maximum or the minimum value of the function, explain that the question asks for the largest or the smallest *y*-value of the function and not for the coordinates of a point. The *x*-coordinate indicates where or when the maximum/minimum value occurs, while the *y*-coordinate identifies the maximum or minimum value of the function. The same maximum or minimum *y*-value may occur at more than one point. All area calculations can be determined using formulas for typical geometric shapes and the answers should be in terms of π.

This lesson contains continuous piecewise functions. Note that in the questions and answers, the written equations include the shared endpoints in each of the domain intervals. (For example, see the answer for question 19.) Students may argue that including the shared endpoint in both intervals creates an equation that is not a function; however this is simply not true. Teachers should explain that, since evaluating both function pieces at their common *x*-value produces the same *y*-value, writing the equation in this way does not contradict the definition of a function which states for each *x*-value, there is only one *y*-value. (The Test Development Committee of the College Board has used this approach on past exams. For an example, see 2006 AB/BC 4 Form B.)

You may wish to support this activity with TI-Nspire™ technology. The TI-Nspire companion document "Analyzing Piecewise Functions," that is posted on the NMSI website, enables students to visualize, explore, and interact with the questions in the lesson. For example, in question 5, students can move a slider across the graph shading the area between the curve and the *x*-axis to help them locate the vertical line that divides the area in half. For questions 14 – 17, a slider is used to investigate the key features of the function. Additional pages display new functions created by transforming $f(x)$ and enables students to see the new bounded regions. A template for entering the piecewise function is provided for question 19. If students are not familiar with entering piecewise functions into the calculator, refer to *Graphing Piecewise Functions* in the NMSI TI-Nspire Skill Builders.

Suggested modifications for additional scaffolding include the following:

5	Draw the isosceles triangle and the trapezoid inside the semicircle to see that the two figures fit into the area of the semicircle with space remaining; therefore, the line $x = k$ must go through the semicircle to divide the area into two equal parts.
6, 9-10	Provide the graphs of the transformations for $g(x)$, $h(x)$, and $r(x)$.
19	Provide the equation for the semi-circular piece and allow the figure on [3, 5] to be described with two linear equations rather than one absolute value equation.
20	Provide the intersection values for the semi-circular piece.
21	Provide the *y*-values of the given points.

NMSI CONTENT PROGRESSION CHART

In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

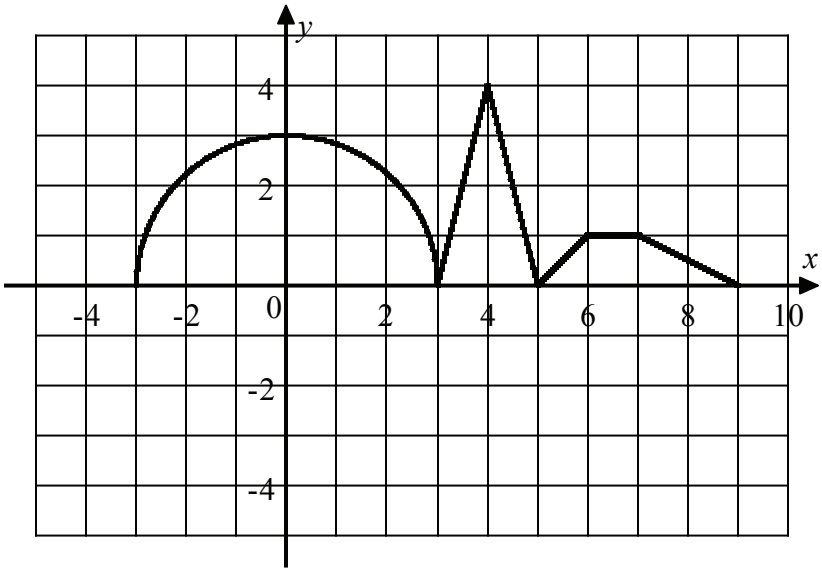
The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of x- and/or y-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of x- and/or y-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of x- and/or y-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	For a continuous piecewise function, determine the x-value given a specific y-value or the y-value given a specific x-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the x-value given a specific y-value or the y-value given a specific x-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the x-value given a specific y-value or the y-value given a specific x-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.
		Write a piecewise linear function based on a graph.	Write a piecewise function based on a graph (may include portions of a circle and linear pieces).	Write a piecewise function based on a graph (may include portions of a circle and/or linear and parabolic pieces).	Write a piecewise function based on a graph (may include portions of conic sections and/or linear, rational, or trigonometric pieces).
		Convert between absolute value functions and piecewise linear functions.	Convert between absolute value functions and piecewise linear functions.	Convert between absolute value functions and piecewise linear functions.	Convert between absolute value functions and piecewise linear function.
			Examine the effects of the transformation $af(x\text{-}c)+d$ on a continuous piecewise function including effects on area.	Examine the effects of the transformation $af(x\text{-}c)+d$ on a continuous or discontinuous piecewise function.	Examine the effects of the transformation $af(b(x\text{-}c))+d$ and $f(\|x\|)$ on a continuous or discontinuous piecewise function.

TEACHER PAGES

Analyzing Piecewise Functions

Use the graph of $y = f(x)$ and the table of values to answer the questions.



x	$f(x)$
-3	0
0	3
3	0
4	4
5	0
6	1
7	1
9	0

1. Classify each of the geometric figures formed between the graph of $f(x)$ and the x-axis.

2. What is the total area enclosed by the graph of $f(x)$ and the x-axis?

3. Give the equation for each line of symmetry, if one exists, on the following intervals of the graph of $f(x)$.

   [–3, 3] _____ [3, 5] _____ [5, 9]_____ [–3, 9]_____

4. For each of the intervals in question 3, how do the lines of symmetry divide the area between the graph of $f(x)$ and the x-axis?

5. A vertical line $x = k$ divides the area enclosed between the graph of $f(x)$ and the x-axis into two equal parts. Which geometric figure does this line intersect?

6. If $g(x) = -f(x)$, determine the total area between the graph of $g(x)$ and the x-axis.

7. Classify each of the geometric figures formed between the graphs of $f(x)$ and $g(x)$.

8. What is the area enclosed by the graphs of $f(x)$ and $g(x)$?

9. If $h(x) = 2f(x)$, what is the total area between the graph of $h(x)$ and the x-axis?
(Hint: The figure between $x = -3$ and $x = 3$ is now half of an ellipse with a semi-major axis of $a = 6$ and a semi-minor axis $b = 3$. The area of an ellipse is calculated by the formula $A = \pi ab$, where a and b are the semi-major and semi-minor axes lengths.)

10. If $r(x) = 4f(x)$, what is the total area between the graph of $r(x)$ and the x-axis?

11. Compare the area between the graph of $f(x)$ and the x-axis to the area between the graph of $p(x)$ and the x-axis, where $p(x) = a\,f(x)$, $a > 0$.

12. What is the total area bounded by the graph of $f(x)$, and the lines $y = -1$, $x = -3$, and $x = 9$?

13. If $q(x) = f(x) + 1$, what is the total area bounded by $q(x)$, the x-axis, $x = -3$, and $x = 9$?

14. On what intervals is $f(x)$ decreasing?

15. On what intervals is $f(x)$ increasing?

16. What is the absolute maximum value of $f(x)$?

17. What is the x-coordinate(s) where the absolute minimum value of $f(x)$ occurs?

18. If $g(x) = 2f(x)$, what is the absolute maximum value?

19. Write the equation for $f(x)$ using five equations.
(Hint: Write an equation involving absolute value for the portion of the function between $x = 3$ and $x = 5$.)

$$f(x) = \begin{cases} \_\_\_\_\_, \_\_\_\_\_ \\ \_\_\_\_\_, \_\_\_\_\_ \\ \_\_\_\_\_, \_\_\_\_\_ \\ \_\_\_\_\_, \_\_\_\_\_ \\ \_\_\_\_\_, \_\_\_\_\_ \end{cases}$$

20. What is x when $f(x) = \frac{1}{2}$? Justify algebraically.

21. Determine the slope at each of the following points on the graph of $f(x)$. If the point is on the semicircle, the slope will be the same as the slope of the tangent line to the semicircle at that point.

(3.5, _____); slope = _____

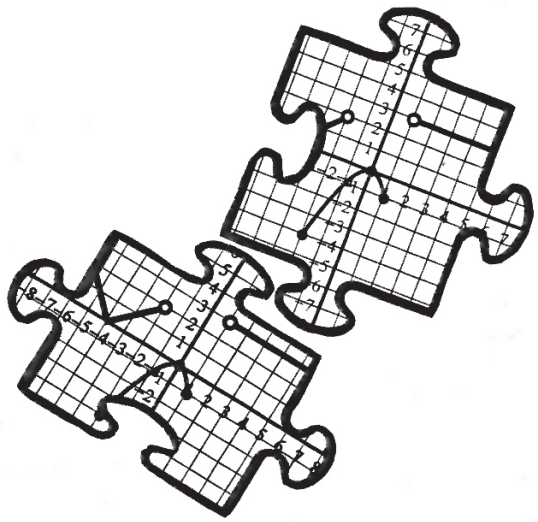
(4.5, _____); slope = _____

(6.213, _____); slope = _____

$\left(7\frac{4}{5}, \_\_\_\_\_\right)$; slope = _____

(–3, _____); slope = _____

(0, _____); slope = _____



Piecewise Puzzle

ABOUT THIS LESSON

This lesson reinforces the skill of writing piecewise functions. Students are given five different sets of domain and range criteria and four quadrant cards with which they must create a piecewise graph that meets each criteria. Once the graph has been created, students sketch the graph and then write the piecewise definition.

OBJECTIVES

Students will

- create a graph of a piecewise function.
- write a piecewise defined function.

LEVEL

Algebra 1 or Math 1, Algebra 2 or Math 2, as a review of graphing and writing functions

MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

TEACHER PAGES

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (⋆) at the end of a specific standard indicates that the high school standard is connected to modeling.

Targeted Standards

F-IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.⋆* *See questions 1-5*

F-BF.1 Write a function that describes a relationship between two quantities.⋆ *See questions 1-5*

Reinforced/Applied Standards

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.⋆* *See questions 1-5*

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.1 Make sense of problems and persevere in solving them. *Students must analyze the domain and range constraints, utilize reasoning skills to consider how the cards can be arranged to fit the constraints, and then create a piecewise definition for the created graph.*

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Write domain and range from a graph
- Write linear, quadratic, and absolute value functions

ASSESSMENTS

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – Algebra 1 Free Response Questions
- Analysis of Functions: Piecewise Graphs – Algebra 1 Multiple Choice Questions
- Analysis of Functions: Piecewise Graphs – Algebra 2 Free Response Questions
- Analysis of Functions: Piecewise Graphs – Algebra 2 Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Applet for exploring parameters in piecewise functions: http://www.mathdemos.org/mathdemos/piecewise/Examples/Java_continuity.html
- NMSI video clips on creating piecewise functions on the TI-84 and TI-Nspire

TEACHING SUGGESTIONS

This lesson is an excellent way to reinforce the concept of writing piecewise functions. Students must use critical thinking skills to create a graph from four quadrant cards to match the given domain and range criteria. It may be necessary to work example as a whole group exercise before having students work with a partner to complete the activity. Strategies that may assist students as they begin to think about each question include: connecting the given domain and range constraints to the end behavior possibilities of the quadrant graphs, noting whether the given domain and range are continuous, and ensuring that the placement of each quadrant graph fulfills the definition of a function. Remind students that each piece can only be used once for each graph but not all the cards are required for each of the questions, the cards can be rotated, and one corner of each card must touch the origin.

To extend this activity, have students identify their own constraints for the domain and range and create graph pieces for classmates to match and write the piecewise function.

You may wish to support this activity with TI-Nspire™ technology. See *Graphing Piecewise Functions* in the NMSI TI-Nspire Skill Builders.

Suggested modifications for additional scaffolding include the following:

1-5 Provide the exact cards needed for each question, mark the pieces to show the top of each card and modify the directions so that students cannot rotate the cards, and have students tape together the pieces for each graph. Provide an equation bank for students to choose from to aid in writing the function.

NMSI CONTENT PROGRESSION CHART

In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

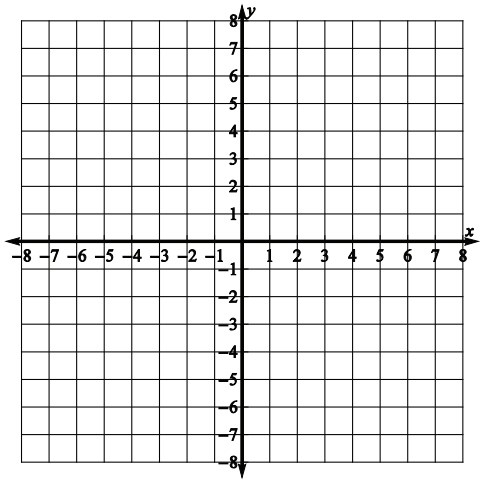
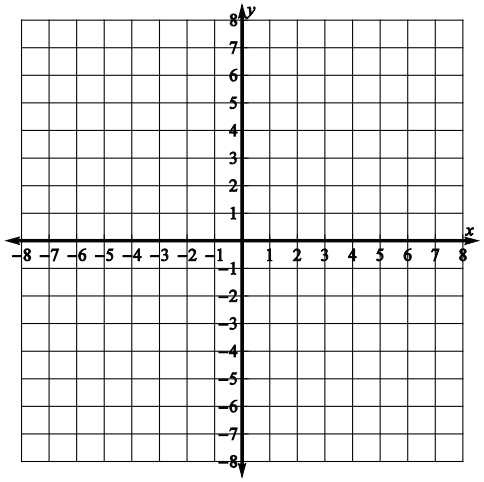
The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

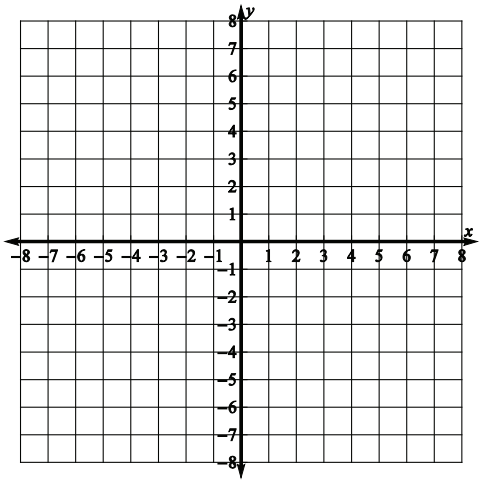
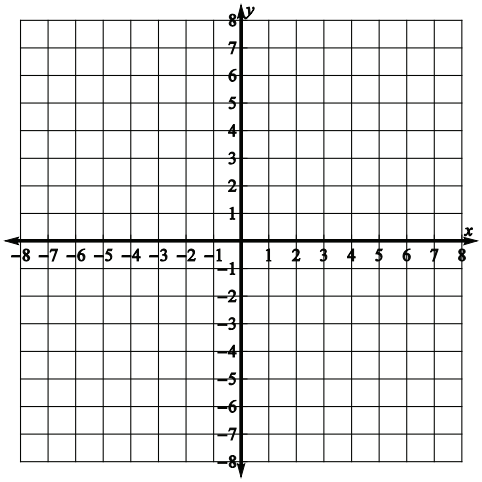
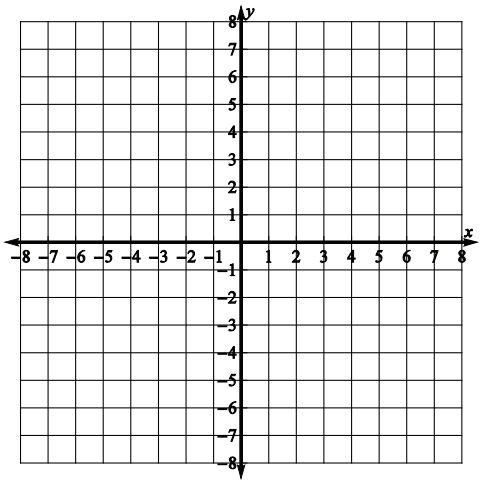
6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	For a continuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.
		Write a piecewise linear function based on a graph.	Write a piecewise function based on a graph (may include portions of a circle and linear pieces).	Write a piecewise function based on a graph (may include portions of a circle and/or linear and parabolic pieces).	Write a piecewise function based on a graph (may include portions of conic sections and/or linear, rational, or trigonometric pieces).

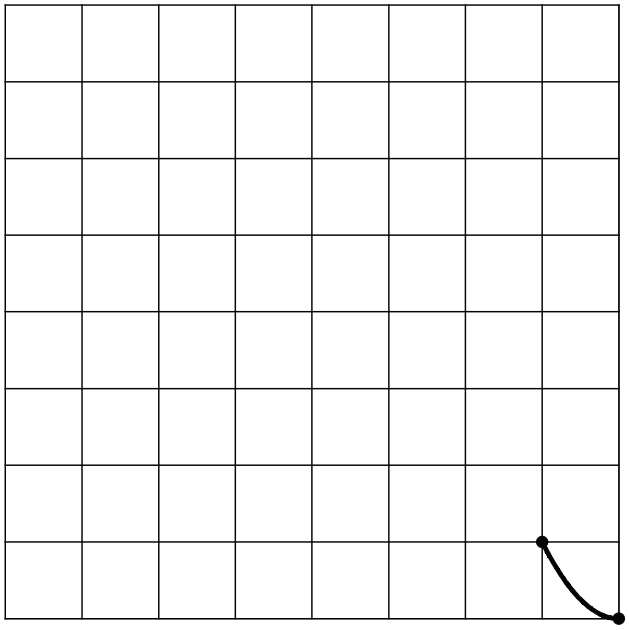
Piecewise Puzzle

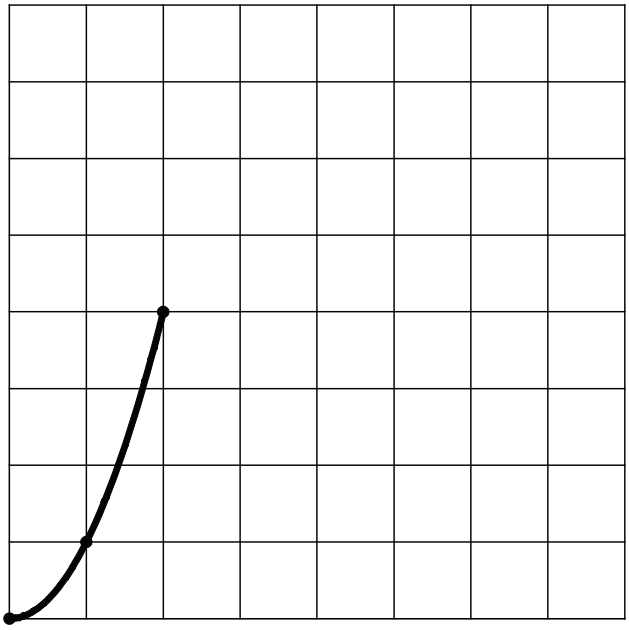
Cut apart the 4 quadrant pieces to make cards that can each be used to represent one quadrant of a coordinate graph. Select one or more of the cards and place them in the appropriate quadrants to create a function that meets the domain and range criteria in each of the five questions. One corner of each card used much touch the origin. Not all of the cards are required for each of the questions. Use the Coordinate Mat to help visualize a solution.

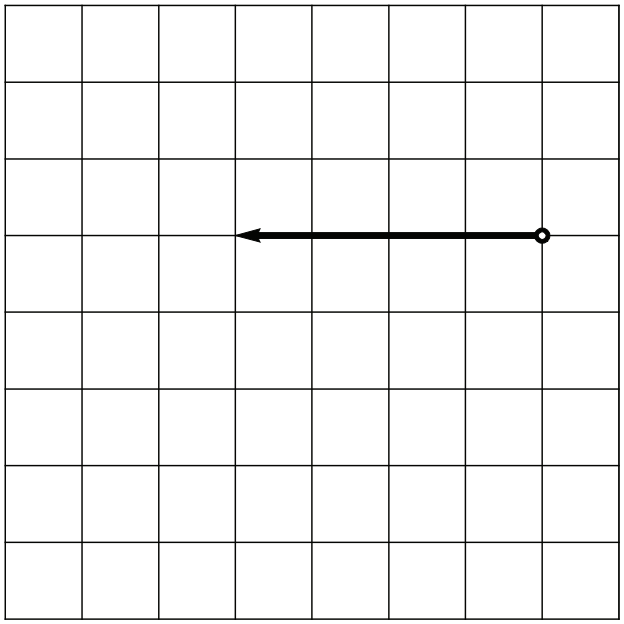
For each question, sketch the solution graph on the recording sheet and write a piecewise definition, including domains, for the function.

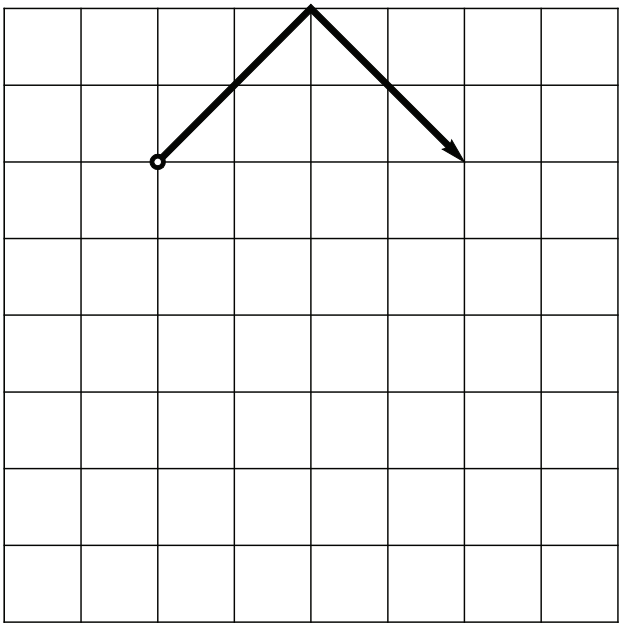
Domain & Range	Piecewise Graph	Piecewise Defined Function
1. Domain: $[-4, \infty)$ Range: $[-1, 2], \{3\}$		$f(x) = \begin{cases} \\ \\ \\ \end{cases}$
2. Domain: $(-\infty, \infty)$ Range: $[-4, \infty)$		$f(x) = \begin{cases} \\ \\ \\ \end{cases}$

3. Domain: $(-\infty, -1)$, $[0, \infty)$ Range: $(-\infty, 4)$, $\{5\}$		$f(x) = \begin{cases} \\ \\ \\ \end{cases}$
4. Domain: $(-\infty, \infty)$ Range: $(-\infty, 4]$		$f(x) = \begin{cases} \\ \\ \\ \end{cases}$
5. Domain: $(-\infty, 0]$, $(1, \infty)$ Range: $[-4, \infty)$		$f(x) = \begin{cases} \\ \\ \\ \end{cases}$









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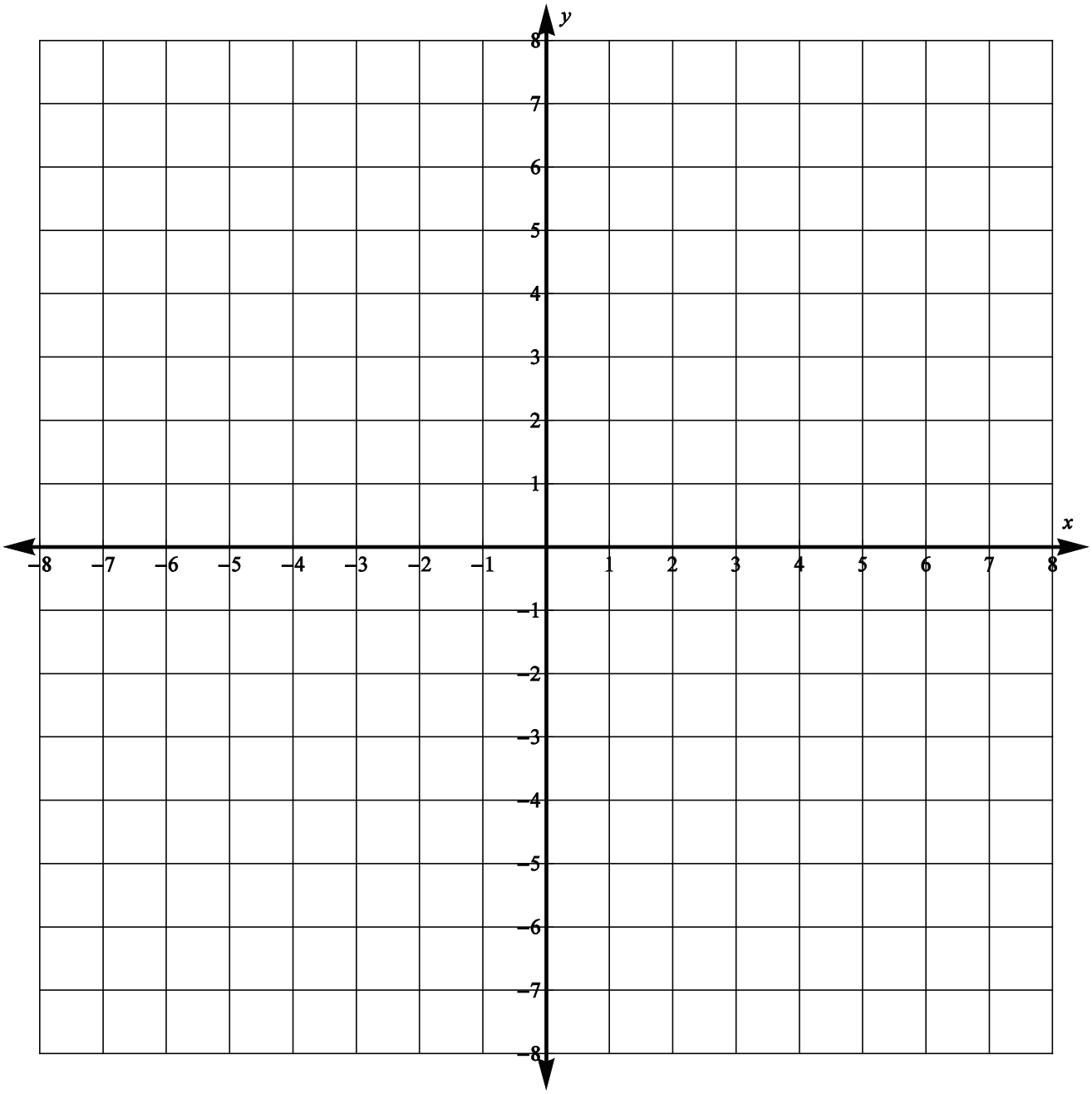

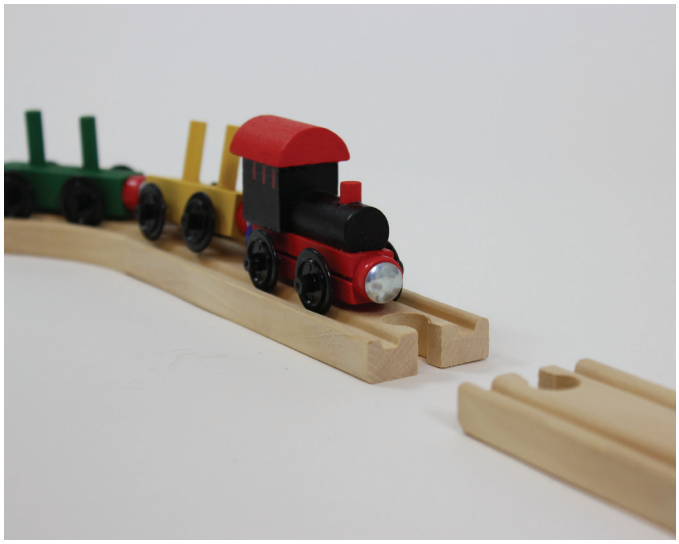

Characteristics of Discontinuous Piecewise Functions

ABOUT THIS LESSON

In this lesson, students analyze the attributes of a discontinuous piecewise function consisting of linear and quadratic pieces. In addition to identifying domain, range, particular independent and dependent function values, intervals of increase or decrease, and absolute maximums and minimums, students are asked to write the equations for both the linear and quadratic pieces. Finally, students translate some of the pieces in order to make the function continuous.

OBJECTIVES

Students will

- analyze the attributes of a discontinuous piecewise function.
- write equations for the linear and parabolic sections of the piecewise function.
- determine transformations of portions of the graph necessary to make the piecewise function continuous.

LEVEL

Algebra 2 or Math 3 in a unit on piecewise-defined functions

Algebra 1 or Math 1 as an extension of a unit including analyzing graphs and applying transformations

MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

TEACHER PAGES

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (★) at the end of a specific standard indicates that the high school standard is connected to modeling

Targeted Standards

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.
Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. ★
See questions 3-6

Reinforced/Applied Standards

F-BF.3: Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.
Include recognizing even and odd functions from their graphs and algebraic expressions for them.
See questions 6b-c, 9-10

F-BF.1: Write a function that describes a relationship between two quantities. ★
See questions 7, 9

A-CED.1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. ★
See question 8

A-REI.4b: Solve equations and inequalities in one variable.
(b) Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
See question 8

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.1: Make sense of problems and persevere in solving them.
In question 9, students must analyze the given constraint of keeping one point fixed and then consider how they might translate the remaining pieces of the function, taking into account which piece to translate first, along with the appropriate distances for each translation.

In question 10, students must analyze the given constraints and consider how they might transform the pieces of the function, taking into account the domain and range of the entire function.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Use basic function vocabulary related to key features of graphs
- Write the equation of a line and a parabola from a graph

ASSESSMENTS

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – Algebra 1 Free Response Questions
- Analysis of Functions: Piecewise Graphs – Algebra 1 Multiple Choice Questions
- Analysis of Functions: Piecewise Graphs – Algebra 2 Free Response Questions
- Analysis of Functions: Piecewise Graphs – Algebra 2 Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Patty paper (optional)
- Applet for exploring parameters in piecewise functions: http://www.mathdemos.org/mathdemos/piecewise/Examples/Java_continuity.html
- NMSI video clips on creating piecewise functions on the TI-84 and TI-Nspire

TEACHING SUGGESTIONS

The intent of this lesson is to connect attributes of previously learned functions with piecewise functions. This type of global lesson encourages retention of important topics by revisiting them each time the student sees a new type of function. If students have been introduced to piecewise functions previously (in another course or earlier in this course), this lesson can serve as independent practice or as a formative assessment.

Several important nuances in analyzing functions are included in this lesson:

1. When the piecewise functions are not continuous, examine the graphs and/or equations carefully to determine which endpoints are included in the domains of particular pieces. (For example, see the answers for questions 4 and 7c.)
2. Once some of the pieces are translated to create a continuous function, the written equations include the shared endpoints in each of the domain intervals. (For example, see the answer for question 9.) Students may argue that including the shared endpoint in both intervals creates an equation that is not a function; however this is simply not true. Teachers should explain that, since evaluating both function pieces at their common x-value produces the same y-value, writing the equation in this way does not contradict the definition of a function which states for each x-value, there is only one y-value. (The Test Development Committee of the College Board has used this approach on past exams. For an example, see 2006 AB/BC 4 Form B.)
3. The maximum or minimum value of the function is only the y-coordinate of a point. The x-value indicates where or when the maximum or minimum occurs, but it is the y-value that is the actual maximum or minimum function value.

In questions 9 and 10 teachers may want to use patty paper or make two transparency copies of the graphs so that the transformations can be seen easily. Students may need help with maintaining the range in question 10. Suggest that they work with the quadratic on (3, 9] first and then move the other pieces to connect endpoints. As an extension of the work in question 10, teachers may want to ask students to write the piecewise function for their new function $f(t)$.

You may wish to support this activity with TI-Nspire™ technology. See *Graphing Piecewise Functions* in the LTF TI-Nspire Skill Builders.

Suggested modifications for additional scaffolding include the following:

7c Complete blanks for all of the domains, as well as the second equation, leaving blanks for only the first and third equations.

9 Provide a graph of a possible solution for the transformation and have the student write the piecewise function of the given transformation.

10 On the graph, draw horizontal and vertical lines to indicate the restriction that the domain and range must remain the same. Provide one possible solution.

NMSI CONTENT PROGRESSION CHART

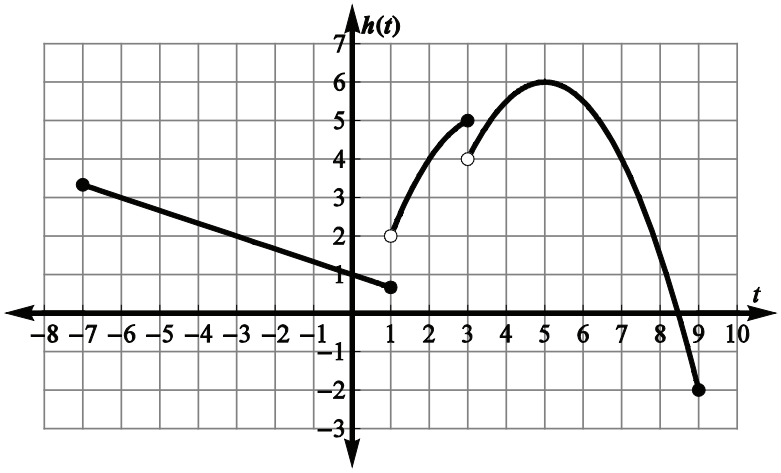
In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	For a continuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.
		Write a piecewise linear function based on a graph.	Write a piecewise function based on a graph (may include portions of a circle and linear pieces).	Write a piecewise function based on a graph (may include portions of a circle and/or linear and parabolic pieces).	Write a piecewise function based on a graph (may include portions of conic sections and/or linear, rational, or trigonometric pieces).
			Examine the effects of the transformation $af(x-c)+d$ on a continuous piecewise function including effects on area.	Examine the effects of the transformation $af(x-c)+d$ on a continuous or discontinuous piecewise function.	Examine the effects of the transformation $af(b(x-c))+d$ and $f(\|x\|)$ on a continuous or discontinuous piecewise function.
				Examine the discontinuity of a piecewise function as well as use transformations to make a function continuous.	Examine the discontinuity of a piecewise function as well as use transformations to make a function continuous.

Characteristics of Discontinuous Piecewise Functions

Use the graph of the piecewise function, $h(t)$, and the table to answer the following questions. The point (3, 5) is <u>not</u> the vertex of a parabola, but the point (5, 6) is the vertex of a parabola.



t	$h(t)$
-6	3
-3	2
2	4
3	5
5	6
7	4
9	-2

1. What is the value of t where $h(t) = 4$?

2. What is the value of $h(3)$?

3. What are the coordinates of the point where $h(t)$ has an absolute maximum value?

4. On what intervals is $h(t)$
   a. increasing?

   b. decreasing?

5. a. What is the domain?

   b. What is the range?

   c. At what values of t is h discontinuous?

6. a. What is the absolute minimum value for $h(t)$?

   b. At what value of t does the absolute minimum value occur for $f(t) = h(t+2)$?

   c. What is the absolute minimum value for $q(t) = h(t) + 2$?

7. a. The portion of the graph on the interval [–7, 1] is linear. What is the equation for this portion of the function?

   b. The portion of the graph on the interval (3, 9] is parabolic. What is the equation for this portion of the function?

   c. The equation of the graph on the interval (1, 3] is $y = -\frac{1}{2}t^2 + \frac{7}{2}t - 1$.

   Combine this equation with the equations from (a) and (b) to form one piecewise rule for $h(t)$.

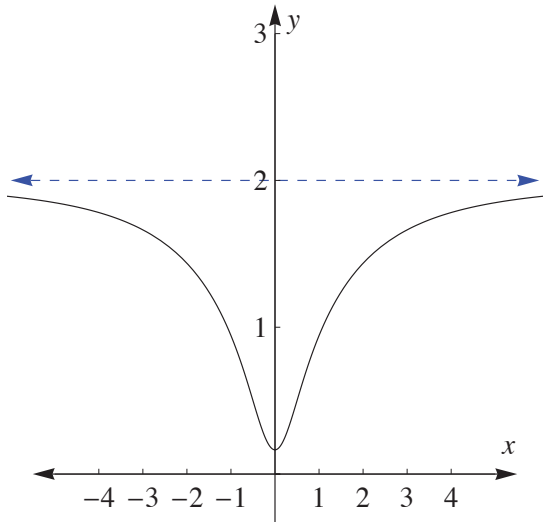
$$h(t) = \begin{cases} \underline{\qquad\qquad\qquad\qquad}, & \underline{\qquad\qquad} \\ \underline{\qquad\qquad\qquad\qquad}, & \underline{\qquad\qquad} \\ \underline{\qquad\qquad\qquad\qquad}, & \underline{\qquad\qquad} \end{cases}$$

8. On what interval is $h(t) < 0$?

9. Keeping $h(0) = 1$, translate the equations on (1, 3] and on (3, 9] to create a new function $g(t)$ that will be continuous on [–7, 9].

$$g(t) = \begin{cases} \underline{\qquad\qquad\qquad\qquad}, & \underline{\qquad\qquad} \\ \underline{\qquad\qquad\qquad\qquad}, & \underline{\qquad\qquad} \\ \underline{\qquad\qquad\qquad\qquad}, & \underline{\qquad\qquad} \end{cases}$$

10. Describe a possible set of transformations of all three pieces of $h(t)$ to create a continuous function $f(t)$ with the same domain and range as h. Explain why f is continuous.



LEVEL

Algebra 2 or Math 3 in a unit on analysis of function graphs or a unit on transformations of functions

Pre-Calculus or Math 4 in a unit reviewing analysis of function graphs or a unit on transformations of functions

MODULE/CONNECTION TO AP*

Analysis of Functions: Piecewise Graphs

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

A Piecewise Function with a Discontinuous Domain

ABOUT THIS LESSON

This lesson extends the concept of piecewise functions to a function with both a domain and range that are discontinuous. Within the lesson, students analyze the attributes of the function and write the equations for linear, parabolic, and sinusoidal sections. In addition, students sketch various transformations of the original function and answer questions about the transformed function. This type of global lesson encourages retention of important topics by revisiting them each time students see a new type of function.

OBJECTIVES

Students will

- analyze the attributes of a discontinuous piecewise function.
- write equations for linear, parabolic, and sinusoidal sections of a piecewise function.
- determine the effect of transformations on a function's graph and attributes.

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (★) at the end of a specific standard indicates that the high school standard is connected to modeling

Targeted Standards

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
See questions 1-9, 10b, 11b, 12b, 13b, 14b, 15b, 17

F-BF.3: Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
See questions 10-15, 18

Reinforced/Applied Standards

F-BF.1: Write a function that describes a relationship between two quantities. ★
See questions 16, 18

F-TF.5: Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★
See questions 16, 18

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.7: Look for and make use of structure. *Students apply their knowledge of the behavior of transformations stated functionally to a given piecewise graph in order to identify function values of the transformed graph.*

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Use basic function vocabulary related to key features of graphs
- Evaluate functions
- Write the equation of a line and a parabola from a graph
- Graph transformations of functions

ASSESSMENTS

The following formative assessment is embedded in this lesson:

- Students engage in independent practice.

The following assessments are located on our website:

- Analysis of Functions: Piecewise Graphs – Algebra 2 Free Response Questions
- Analysis of Functions: Piecewise Graphs – Algebra 2 Multiple Choice Questions
- Analysis of Functions: Piecewise Graphs – Pre-Calculus Free Response Questions
- Analysis of Functions: Piecewise Graphs – Pre-Calculus Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Patty paper (optional)
- Applet for exploring parameters in piecewise functions: http://www.mathdemos.org/mathdemos/piecewise/Examples/Java_continuity.html
- NMSI video clips on creating piecewise functions on the TI-84 and TI-Nspire

TEACHING SUGGESTIONS

The graph presented in this lesson is a piecewise function with a discontinuous domain and range; however, the questions in the lesson could be used with any graph because these questions involve the attributes of a function that should be emphasized throughout the year. Connections are made between transformations, continuity, piecewise functions, roots, maximum and minimum values. Provide class time for this lesson so that the students can work in groups. The questions encourage mathematical conversations

When piecewise functions are not continuous, examine the graphs and/or equations carefully to determine which endpoints are included in the domains of particular pieces. For adjacent pieces of the function that are continuous, the questions and answers include the shared endpoints in each of the domain intervals. (For example, see the answers for questions 4, 16, and 18.) Students may argue that including the shared endpoint in both intervals creates an equation that is not a function; however this is simply not true. Teachers should explain that, since evaluating both function pieces at their common x-value produces the same y-value, writing the equation in this way does not contradict the definition of a function which states for each x-value, there is only one y-value. (The Test Development Committee of the College Board has used this approach on past exams. For an example, see 2006 AB/BC 4 Form B.)

Question 16 requires students to write the equation for the transformation of a trigonometric function, and question 17 requires students to determine a zero of the trigonometric function.

If this lesson is assigned early in the year, teachers may choose to either (1) change the directions to ask students to write the equation over the domain [–2, 6], (2) provide the student with the equation for the transformed trigonometric function, or (3) replace the graph with the one provided on the last page of the answer key, which uses $y = \frac{5}{4}(x+6)^2 - 2$ instead of the trigonometric function. A blank grid is provided in the answer section for creating additional replacement graphs of the teacher's choice.

To create an extension for this lesson, ask the students to draw their own graph and to write ten questions referencing their graph. Have the students exchange lessons for homework. During the next class period, the student who authored the question should grade the other student's work.

You may wish to support this activity with TI-Nspire™ technology. See *Graphing Piecewise Functions* in the NMSI TI-Nspire Skill Builders.

Suggested modifications for additional scaffolding include the following:

5	Draw the line $y = x$ on the graph.
10-15	Use patty paper to illustrate each of the transformations.
16	Modify the question to provide the cosine piece for the interval $[-6, -4]$, or replace this piece of the graph with a parabola.
17	Allow the student to use a calculator to identify the root on the interval $[-6, -4]$.
18a	Add the following statement to the question: "Hint: this transformation reflects the graph for positive x-values across the y-axis, creating a mirror image of the 1st quadrant graph in the 2nd quadrant."
18b	Provide a template for the piecewise function, filling in each of the domains and selected sections of the function for the student to use as a model. See the answer key for format.

NMSI CONTENT PROGRESSION CHART

In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	Using a piecewise graph involving a real-world application (including rate graphs), identify and interpret the meaning of *x*- and/or *y*-coordinates, domain and/or range, maximum and/or minimums, and intervals where the graph is increasing and/or decreasing.	For a continuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.	For a continuous or discontinuous piecewise function, determine the *x*-value given a specific *y*-value or the *y*-value given a specific *x*-value, the domain and/or range, the maximums and/or minimums, and the intervals where the graph is increasing and/or decreasing.
		Write a piecewise linear function based on a graph.	Write a piecewise function based on a graph (may include portions of a circle and linear pieces).	Write a piecewise function based on a graph (may include portions of a circle and/or linear and parabolic pieces).	Write a piecewise function based on a graph (may include portions of conic sections and/or linear, rational, or trigonometric pieces).
			Examine the effects of the transformation $af(x-c)+d$ on a continuous piecewise function including effects on area.	Examine the effects of the transformation $af(x-c)+d$ on a continuous or discontinuous piecewise function.	Examine the effects of the transformation $af(b(x-c))+d$ and $f(\|x\|)$ on a continuous or discontinuous piecewise function.

TEACHER PAGES

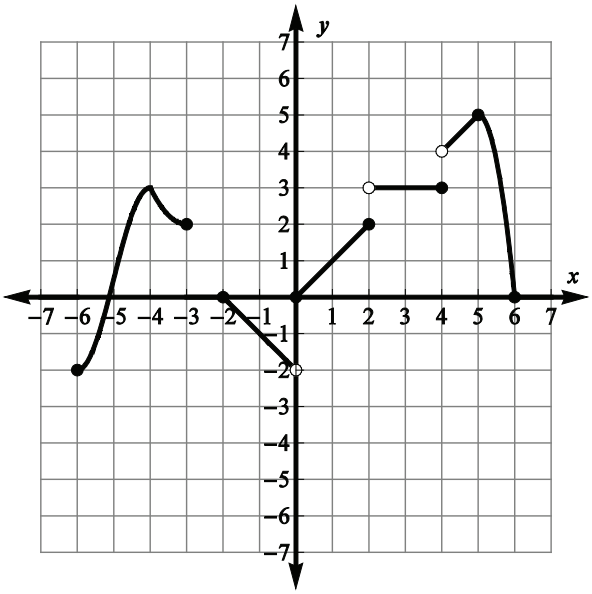
(continued from previous page)

6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	**Pre-Calculus Skills/Objectives**
				Examine the discontinuity of a piecewise function as well as use transformations to make a function continuous.	Examine the discontinuity of a piecewise function as well as use transformations to make a function continuous.
					Determine the concavity of a continuous or discontinuous piecewise function.
					Determine $g(a)$, given g defined as a transformation of f.

A Piecewise Function with a Discontinuous Domain

Use the graph of $f(x)$ and the table to answer the questions.

x	$f(x)$
–6	–2
–4	3
–2	0
–1	–1
1	1
2	2
3	3
4	3
4.5	4.5
5	5
6	0



1. What is the range of $f(x)$?

2. What is the domain?

3. On what intervals is $f(x)$ decreasing?

4. On what intervals will the following statements be true?
   a. As x increases, y increases.

   b. As x increases, y is constant.

   c. As x increases, y increases at a constant rate.

5. For what values of x is $f(x) > x$?

6. What is the absolute maximum value for $f(x)$?

7. What are the coordinates of the point where the global minimum value of $f(x)$ occurs?

8. What is the absolute maximum value over the interval $-6 \le x \le -3$?

9. For what values of x, where $x > 0$, is $f(x)$ concave down?

10. Let $g(x) = f(-x)$
    a. What is the value of $g(-4)$? of $g(2)$?

    b. For what value of x does the maximum value of $g(x)$ occur?

    c. Describe how the graph of $g(x)$ is a transformation of the graph of $f(x)$.

11. Let $g(x) = -f(x)$
    a. What is the value of $g(-2.5)$? of $g(4)$?

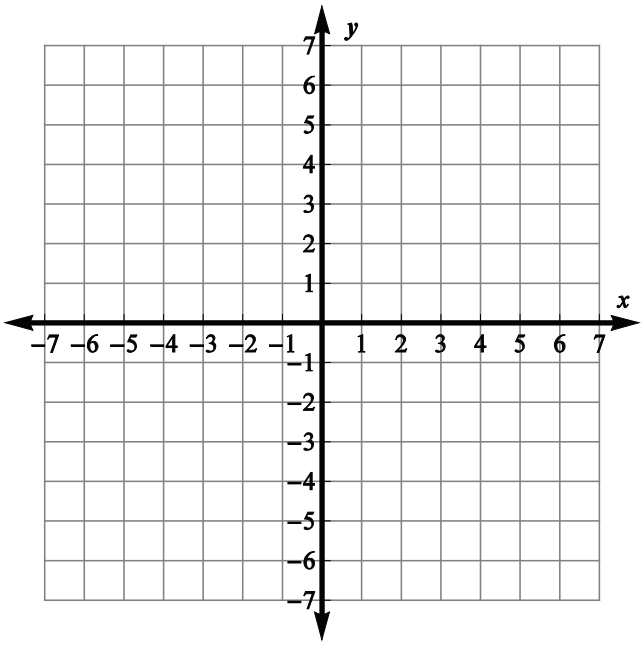
    b. What is the minimum value of $g(x)$?

    c. Describe how the graph of $g(x)$ is a transformation of the graph of $f(x)$.

12. Let $g(x) = f(2x)$
    a. What is the value of $g(-2)$? of $g(-1)$?

    b. What is the slope of $g(x)$ on the interval [0, 1]?

    c. Sketch a graph of $g(x)$.

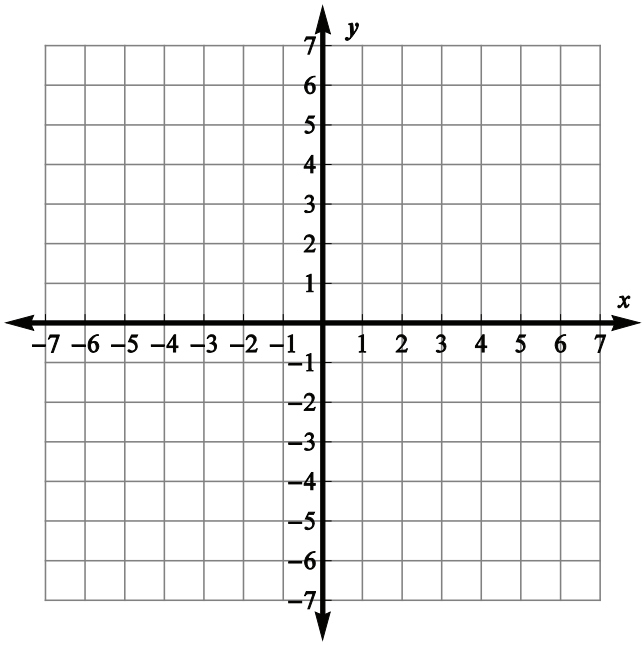


13. Let $g(x) = f(x-1)$

a. What is the value of $g(0)$? of $g(1)$?

b. On what intervals is $g(x)$ increasing?

c. Sketch a graph of $g(x)$.



14. Let $g(x) = f(x+3)$

a. What is the value of $g(-2)$? of $g(3)$?

b. If $x > 0$, where is $g(x)$ concave down?

c. Describe how the graph of $g(x)$ is a transformation of the graph of $f(x)$.

15. Let $g(x) = f(x) + 3$

a. What is the value of $g(-2)$? of $g(3)$?

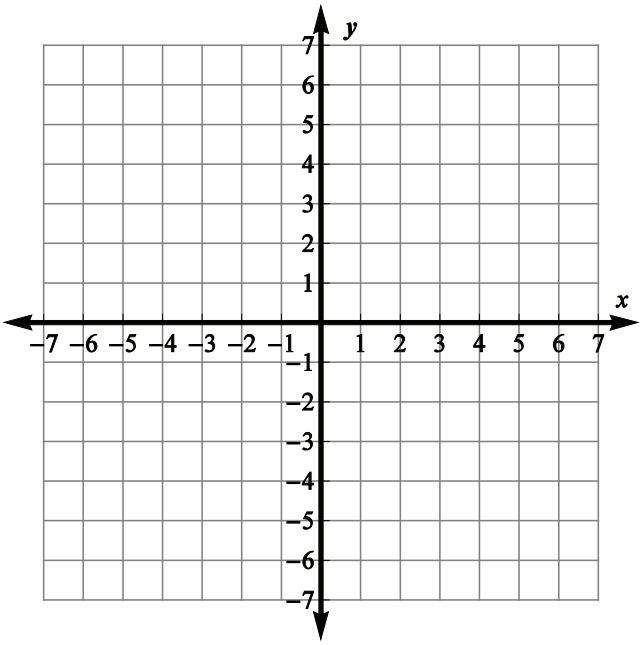
b. What is the y-intercept of $g(x)$?

c. Describe how the graph of $g(x)$ is a transformation of the graph of $f(x)$.

16. Write a piecewise function to describe $f(x)$ over the domain [–6, –3] and [–2, 6].
The portion of the graph on the interval [–6, –4] should be modeled as a cosine function with a maximum at (–4, 3) and a minimum at (–6, –2).
The section on [–4, –3] should be modeled as a parabola with its vertex at (–3, 2).
The section on [5, 6] is a parabola with its vertex at (5, 5).

17. What are the roots of $f(x)$?

18. Let $h(x) = f(|x|)$.
    a. On the grid provided, graph $h(x)$.
    b. Write a piecewise function to describe $h(x)$ using as few equations as possible. Hint: Use absolute values to decrease the number of required equations.



Introduction to the NMSI Mathematics Multiple Choice Quizzes

The National Math and Science Initiative multiple choice questions are modeled after multiple choice questions on the AP* Calculus and Statistics exams. The questions are assigned a course-level designation based on an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The grade-level multiple choice quizzes for sixth grade through pre-calculus assess the skills and concepts introduced in each module. These quizzes reflect the module's Content Progression Chart, which outlines the mathematics imbedded in the activities for each grade level, and the Concept Development Chart, which provides examples of how those concepts or skills might be assessed. Additionally, the quizzes are directly linked to the NMSI posttests for each grade level. Once students have completed the activities, teachers may use the quiz questions to determine student understanding and to prepare students for the level of rigor on the posttests.

When scoring the multiple choice questions, teachers should remember that the quizzes are intended to model the rigor of questioning on AP exams. A suggested scoring guideline, which is also included with the rationales for each quiz, is:

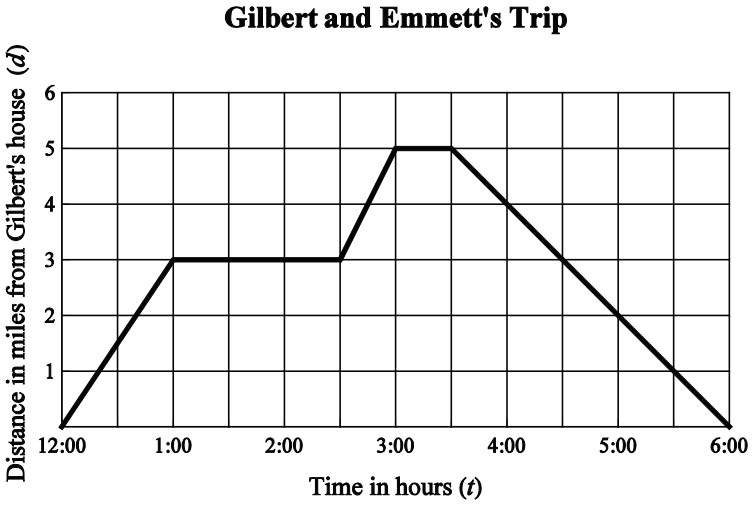
Percent Correct	Grade
0 – 29	50
30 – 49	60
50 – 59	70
60 – 69	80
70 – 79	90
80 – 100	100

All of these materials – lessons and activities with answer keys, grade-level quizzes with rationales, and free response questions with scoring rubrics and student samples – are available for each module on the NMSI website.

Sample Quiz Questions

Use the following information and graph to answer questions 1 and 2
(6th Grade Module 4 Questions 5 and 9).

Gilbert and Emmett ride their bicycles along Main Street from Gilbert's house to the movie theater to watch the afternoon matinee. After the movie, they continue in the same direction down Main Street to get a snack at the ice cream shop. They then return along their same path to Gilbert's house. The graph shows their distance from Gilbert's house at any time during the afternoon.



1. 6th Grade Module 4 Question 5
   How many hours were the boys actually riding their bicycles?

   A. 2 hours

   B. 3 hours

   C. 3.5 hours

   D. 4 hours

   E. 6 hours

2. 6th Grade Module 4 Question 9
   Which value completes the table?

   A. 2

   B. 3

   C. 4

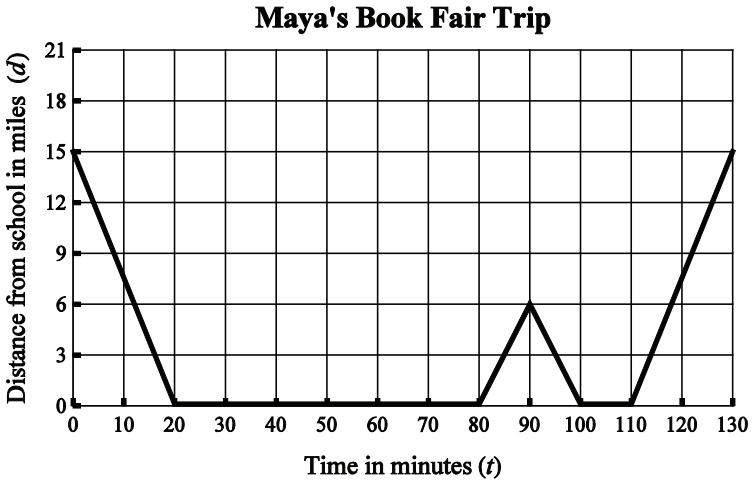
   D. 5

   E. 6

Time (hours)	Distance from Gilbert's house (miles)
3:30	5
4:00	4
5:00	?

3. 7th Grade Module 4 Question 3

Maya and her mother drive from their home to Maya's school on Saturday to attend a book fair. As they begin their trip back home, Maya realizes that she left her backpack at the school. Her mom drives her back to the school, where Maya searches for and finds her backpack. They then drive directly home from the school. The graph shows their distance from the school at any time during the trip.

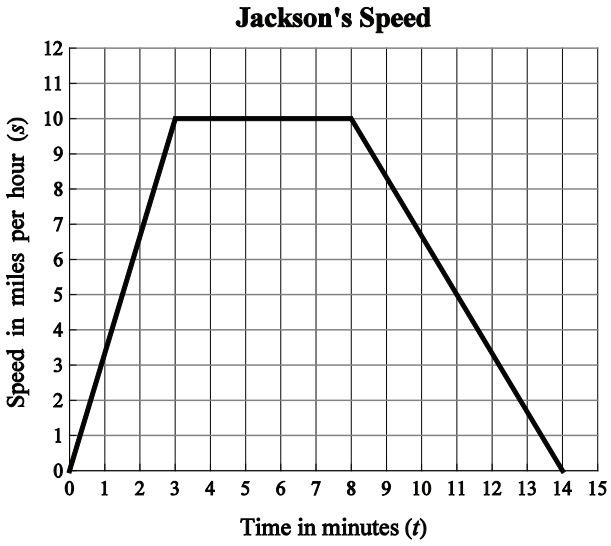


On which time interval(s) is their distance from the school increasing?

I. 0 – 20 minutes
II. 80 – 90 minutes
III. 110 – 130 minutes

A. I only
B. II only
C. III only
D. I and III only
E. II and III only

4. 7th Grade Module 4 Question 9
Jackson rides his bicycle to school along a straight road. The graph shows his speed at any time during his trip to school.



On which of the following time intervals is Jackson riding at a constant speed for the entire interval?

A. 0 minutes to 3 minutes

B. 4 minutes to 6 minutes

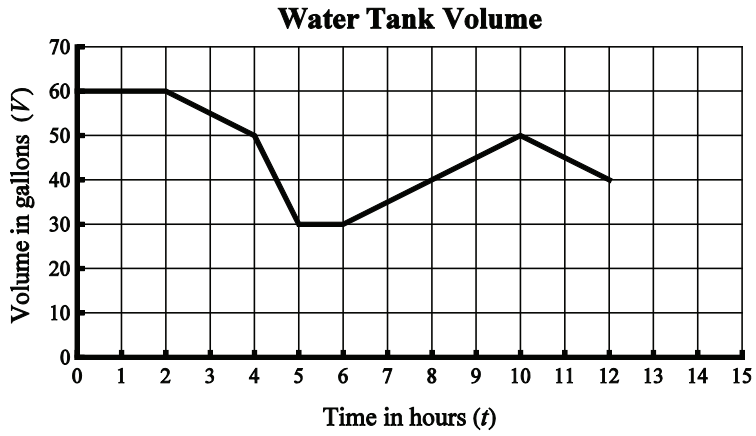
C. 5 minutes to 9 minutes

D. 7 minutes to 10 minutes

E. 8 minutes to 14 minutes

5. Algebra 1 Module 4 Question 2

A water tank collects and stores water used for gardening. At any given time, water may be flowing into or out of the tank. The graph shows the volume of water in the tank, in gallons, over a period of 12 hours.



Which of the following is the piecewise function for the graph on the time interval $0 \le t \le 5$?

A. $V = \begin{cases} 60, & 0 \le t \le 2 \\ -5(t-4)+50, & 2 \le t \le 4 \\ -20(t-4)+50, & 4 \le t \le 5 \end{cases}$

B. $V = \begin{cases} 60, & 0 \le t \le 2 \\ -5(t-2)+60, & 2 \le t \le 4 \\ -10(t-5)+30, & 4 \le t \le 5 \end{cases}$

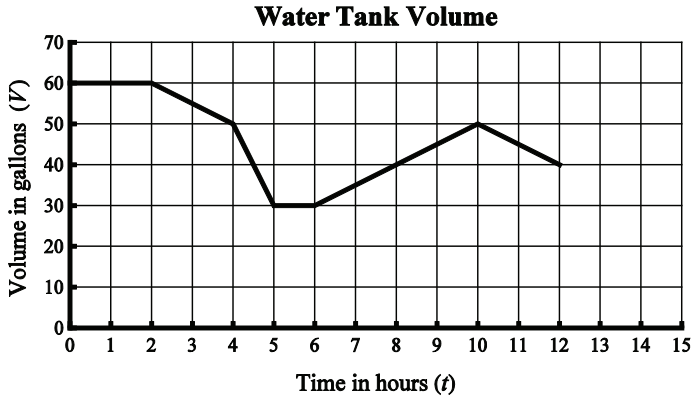
C. $V = \begin{cases} 60t, & 0 \le t \le 2 \\ -5(t-2)+60, & 2 \le t \le 4 \\ -20(t-5)+30, & 4 \le t \le 5 \end{cases}$

D. $V = \begin{cases} t+60, & 0 \le t \le 2 \\ -5(t-4)+50, & 2 \le t \le 4 \\ -20(t-4)+50, & 4 \le t \le 5 \end{cases}$

E. $V = \begin{cases} -t+60, & 0 \le t \le 2 \\ -5t+70, & 2 \le t \le 4 \\ -10t+80, & 4 \le t \le 5 \end{cases}$

6. Algebra 1 Module 4 Question 5

A water tank collects and stores water used for gardening. At any given time, water may be flowing into or out of the tank. The graph shows the volume of water in the tank, in gallons, over a period of 12 hours.



If the graph were extended for the time interval, $12 \le t \le 15$, using the equation, $V = 8(t-12)+40, 12 \le t \le 15$, what would be the volume of water at the end of the time period?

A. 48 gallons

B. 52 gallons

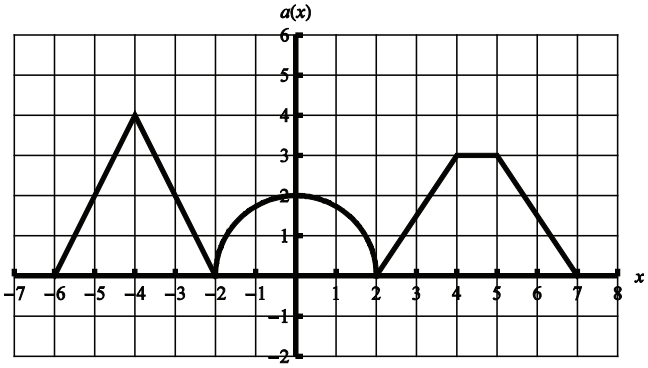
C. 56 gallons

D. 60 gallons

E. 64 gallons

7. Geometry Module 4 Question 2

The piecewise function $a(x)$ consists of linear and semi-circular pieces.



What is the area of the region bounded by $a(x)$ and the x-axis?

A. $17+2\pi$ square units

B. $17+4\pi$ square units

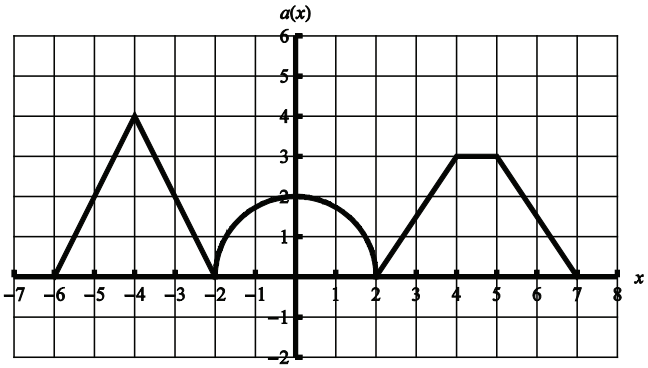
C. $26+2\pi$ square units

D. $34+2\pi$ square units

E. $34+4\pi$ square units

8. Geometry Module 4 Question 6

The piecewise function $a(x)$ consists of linear and semi-circular pieces.



Which transformation(s) would <u>not</u> affect the area bounded by $a(x)$, the x-axis, $x = -6$, and $x = 7$?

I. $|a(x)|$

II. $-a(x)$

III. $a(x) - 4$

A. I only

B. II only

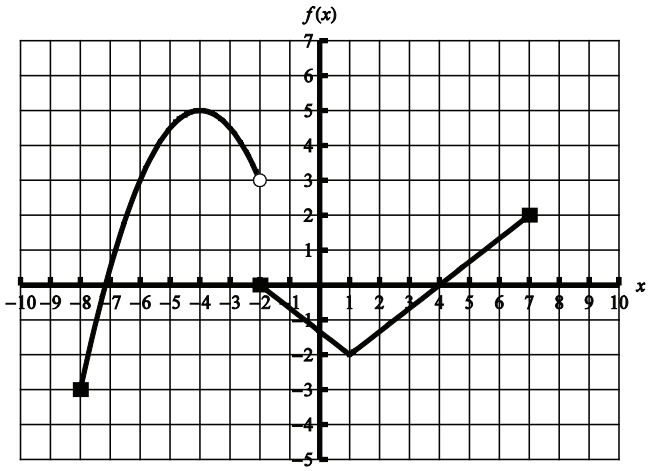
C. I and II only

D. II and III only

E. I, II, and III

9. Algebra 2 Module 4 Question 3

The piecewise-defined function $f(x)$ consists of a portion of a parabola with a vertex at $(-4, 5)$ and a portion of an absolute value function, as shown on the graph.



What is the equation of the absolute value piece of the function on the interval $[-2, 7]$?

A. $f(x) = -\frac{3}{2}|x+1| - 2$

B. $f(x) = -\frac{2}{3}|x-1| - 2$

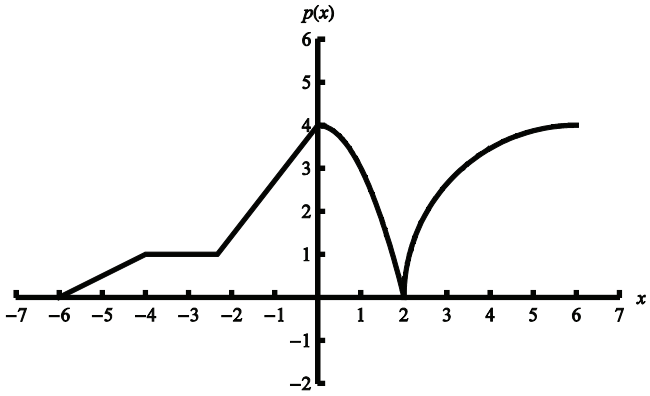
C. $f(x) = \frac{2}{3}|x-1| - 2$

D. $f(x) = \frac{2}{3}|x+1| - 2$

E. $f(x) = \frac{3}{2}|x-1| - 2$

10. Algebra 2 Module 4 Question 8 (Calculator Allowed)
The piecewise function $p(x)$ is defined and graphed as shown.

$$p(x)=\begin{cases} \frac{1}{2}(x+6), & [-6,-4] \\ 1, & \left[-4,-\frac{7}{3}\right] \\ \frac{9}{7}\left(x+\frac{7}{3}\right)+1, & \left[-\frac{7}{3},0\right] \\ -x^2+4, & [0,2] \\ \sqrt{16-(x-6)^2}, & [2,6] \end{cases}$$

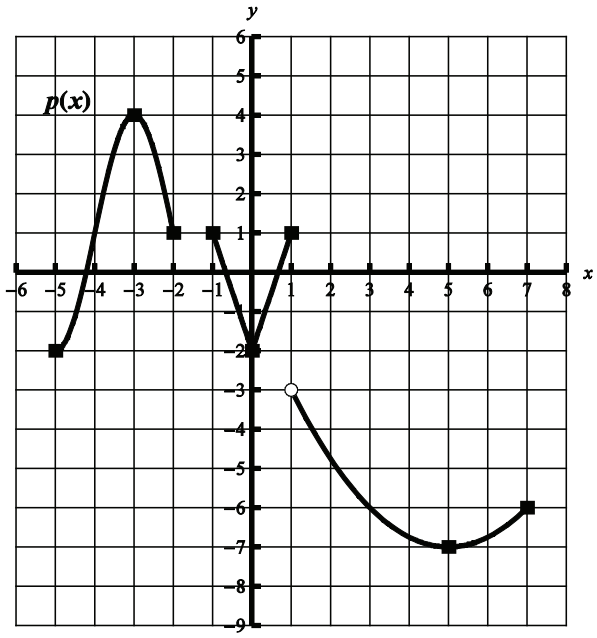


The value of $p(x)=2$ when $x=2.536$. At what other values of x does $p(x)=2$?

A. -1.556 and 1.414
B. -1.556 and 1.489
C. -1.489 and 1.489
D. -1.414 and 1.414
E. -1.556 and 2.553

11. Pre-Calculus Module 4 Question 2

The piecewise function $p(x)$ consists of portions of a sine function, an absolute value function, and a parabolic function.



Which of the following could model $p(x)$ on the interval $[-5, -2]$?

A. $p(x) = -3\sin\left(\frac{2\pi}{3}(x+2)\right)+1$

B. $p(x) = -3\sin\left(\frac{\pi}{3}(x+3)\right)+1$

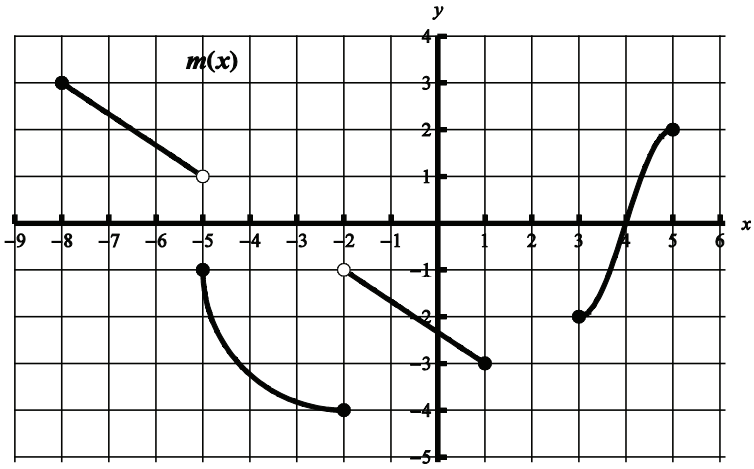
C. $p(x) = 3\sin\left(\frac{\pi}{2}(x+3)\right)+4$

D. $p(x) = 3\sin\left(\frac{\pi}{2}(x+4)\right)+1$

E. $p(x) = 3\sin\left(\frac{2\pi}{3}(x+4)\right)+1$

12. Pre-Calculus Module 4 Question 8

The piecewise function $m(x)$ consists of two linear segments and portions of a parabolic function and a sinusoidal function.



If $b(x) = -2m(x)$, for which of these values of x does $b(x) = 2$?

A. -5

B. -2

C. -1

D. 3

E. 5

Selected Rationales

1. 6th Grade Module 4 Question 5
   A. Student chooses the number of hours that the boys were not moving.
   B. Student selects the time when the boys reached the maximum distance from Gilbert's house.
   C. Student identifies the time elapsed from the end of the movie until the boys arrived back at Gilbert's house.
   D. Correct. Student adds the length of the time periods when the graph is not horizontal.
   E. Student chooses the total time shown on the graph.

2. 6th Grade Module 4 Question 9
   A. Correct. Student selects the distance from Gilbert's house at 5:00.
   B. Student continues the pattern shown in the table.
   C. Student uses the previous number in the table.
   D. Student selects the maximum distance from home or thinks 4:00 corresponds to 4 miles so 5:00 must correspond to 5 miles.
   E. Student selects the ending time.

3. 7th Grade Module 4 Question 3
   A. Student chooses an interval where the distance from home, rather than school, is increasing.
   B. Student chooses only one of the correct intervals where the distance from school is increasing.
   C. Student chooses only one of the correct intervals where the distance from school is increasing.
   D. Student chooses the intervals for the trips between home and school.
   E. Correct. Student selects both intervals where the distance from school is increasing as time increases.

4. 7th Grade Module 4 Question 9
   A. Student chooses an interval with a speed that is increasing at a constant rate.
   B. Correct. Student chooses an interval where the graph is horizontal, indicating a constant speed.
   C. Student selects an interval where the speed is constant for only part of the time.
   D. Student selects an interval where the speed is constant for only part of the time.
   E. Student chooses an interval with a speed that is decreasing at a constant rate.

5. Algebra 1 Module 4 Question 2

A. Correct. Student recognizes the equations for the horizontal line segment and for the two oblique segments that share the point $(4, 50)$.

B. Student selects an incorrect slope for the segment on $[4, 5]$.

C. Student chooses an incorrect equation for the horizontal line segment.

D. Student selects an incorrect equation for the horizontal line segment.

E. Student chooses incorrect equations for the horizontal line segment and for the segment on $[4, 5]$.

6. Algebra 1 Module 4 Question 5

A. Student adds $8+40$ from the given equation.

B. Student adds $12+40$ from the given equation.

C. Student computes $40+2(8)$.

D. Student chooses the value of V at time 0 as the ending value as well.

E. Correct. Student substitutes 15 for t in the given equation and evaluates V.

7. Geometry Module 4 Question 2

A. Correct. Student calculates the area of the triangle, semi-circle, and trapezoid, using $A=\frac{1}{2}(4)(4)+\frac{1}{2}\pi\left(2^2\right)+\frac{1+5}{2}(3)$.

B. Student does not multiply by $\frac{1}{2}$ in the area of the semi-circle or calculates the circumference.

C. Student does not multiply by $\frac{1}{2}$ in the trapezoid area formula.

D. Student does not multiply by $\frac{1}{2}$ in either the triangle or the trapezoid area formulas.

E. Student does not multiply by $\frac{1}{2}$ in any of the area formulas.

8. Geometry Module 4 Question 6

   I. True. Since the function is entirely above the x-axis, the absolute value transformation does not affect the graph.

   II. True. Reflecting the function across the x-axis merely moves the area, but does not change its value.

   III. False. After translating the function 4 units down, the bounds for the area include the vertical lines, and the area is a different shape and size from the areas described in I and II.

   A. Student recognizes that this transformation does not change the function, but fails to consider the other transformations.

   B. Student recognizes that this transformation reflects the function across the x-axis and does not affect the bounded area, but student fails to consider the other transformations.

   C. Correct. Student recognizes that $|a(x)|$ makes no change to the function, and $-a(x)$ merely reflects the function across the x-axis, while $a(x)-4$ creates an area of a different shape whose bounds include the vertical lines, as well as $a(x)$ and the x-axis.

   D. Student thinks that both of these transformations create the same bounded area below the x-axis.

   E. Student thinks that vertically translating the function so that it is entirely below the x-axis will not affect the area of the bounded region.

9. Algebra 2 Module 4 Question 3

   A. Student chooses an incorrect scale factor and an incorrect horizontal translation.

   B. Student selects the wrong sign on the scale factor or calculates the slope on the first section of the absolute value piece of the function.

   C. Correct. Student recognizes that the parent absolute value function has been scaled by a factor of $\frac{2}{3}$ and has been translated 1 unit right and 2 units down.

   D. Student chooses an incorrect horizontal translation.

   E. Student selects an incorrect scale factor.

10. Algebra 2 Module 4 Question 8

   A. Correct. Student uses the "calculate intersection" function of the graphing calculator to determine the intersections of $p(x)$ with $y=2$.

   B. Student calculates the intersection with the line segment but then traces to estimate the value on the parabola.

   C. Student traces to estimate the values on the line segment and the parabola.

   D. Student calculates the intersection with the parabola but ignores the domain restrictions.

   E. Student calculates the intersection with the line segment but then traces to estimate the value on the circle, which was actually given in the stem of the question.

11. Pre-Calculus Module 4 Question 2

A. Student calculates an incorrect value for *b*, using the interval $[-5, -2]$ as the length of one period.

B. Student calculates an incorrect value for *b,* using the interval $[-5, -2]$ as half the length of one period and also selects an incorrect horizontal translation.

C. Student chooses the coordinates of the maximum point as the horizontal and vertical translations.

D. Correct. Student uses the point $(-4, 1)$ as the new origin point of a parent sine function then calculates the amplitude of 3 and the *b*-value of $\frac{\pi}{2}$, using the interval $[-5, -3]$ as half the length of one period.

E. Student calculates an incorrect value for *b*.

12. Pre-Calculus Module 4 Question 8

A. Correct. Student recognizes that $b(x) = 2$ when $m(x) = -1$ and chooses -5 as one of the locations where this is true.

B. Student multiplies $-2(2)$ and looks for a location where $m(x) = -4$.

C. Student selects the value of $m(x)$ that will make $b(x) = 2$.

D. Student chooses a value of *x* where $m(x) = -2$.

E. Student chooses a value of *x* where $m(x) = 2$.

Introduction to the NMSI Mathematics Free Response Questions

The National Math and Science Initiative free response questions are modeled after free response questions on the AP* Calculus and Statistics exams. The questions are assigned a course-level designation based on an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The free response rubric is a guide to assist the reader, not a detailed solution to the question. Sometimes a method is outlined in the rubric, but another more efficient method may work as well. A student's correct solution may earn all of the points to be awarded for a particular part of the question, even though the approach does not match the one shown in the rubric. The rubric shows "a way" to work the problem, not "the way" to work the problem.

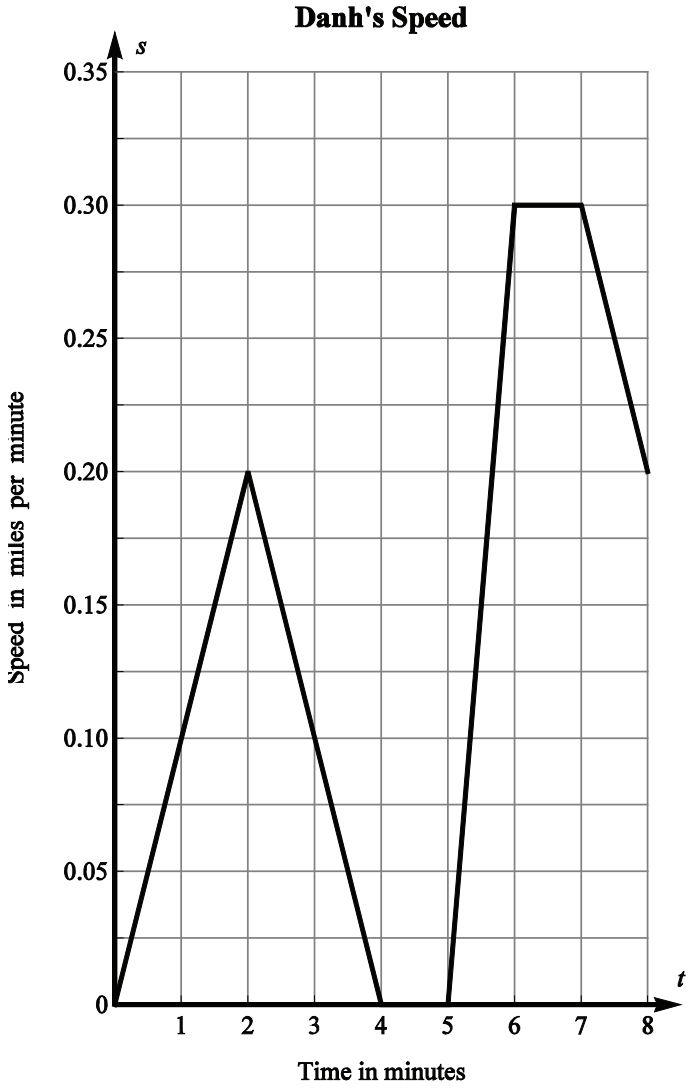
When scoring the free response questions, teachers should practice "reading with" a student's error. This statement means that the student is penalized for the error when it first occurs, but the reader then follows the student's process for full credit in subsequent parts of the question, even when the student continues to use the results of the earlier error. For the free response, the reader should be in the mindset of awarding points, not taking them away. Students start at 0 and can earn up to 9 points rather than starting at 9 points and losing points.

Student directions for the free response questions include the following:

- All work for a given part of a question must be shown in the space provided.
- Answers do not need to be simplified completely; however, when calculating approximate answers, do not round intermediate values. Your final answers should be accurate to three places after the decimal point.
- Questions that contain units require units in the answers.
- The setup for all mathematical computations and equations is required using mathematical notation rather than calculator syntax. Intermediate calculations do not have to be shown when determining:
  - the answer to basic arithmetic computations;
  - the zeroes of a function;
  - the maximum/minimum of a function;
  - the intersection point between two functions;
  - a regression equation.
- Part A and B are given equal weight, but parts of a particular question are not necessarily given equal weight.
- During the timed portion for Part A, you may work only on Part A. A calculator may be used on Part A only.
- During the timed portion for Part B, in addition to working on the question in Part B, you may continue to work on Part A without a calculator.

Grade 6 2010 Free Response Question - Calculator Not Allowed

Danh rides his bicycle along a straight road starting from home at $t = 0$ minutes. The speed graph that consists of six line segments shows his speed, in miles per minute, for the first eight minutes of his trip.



time in minutes	speed in $\frac{\text{miles}}{\text{minute}}$
0	0
2	0.2
4	0
5	0
6	0.3
7	0.3
8	0.2

(a) Danh's speed is increasing between 0 and 2 minutes. Between what other times is Danh's speed increasing?

Between what times is Danh's speed decreasing?

(b) On this graph, the point (4, 0) indicates that at 4 minutes, Danh's speed is 0 miles per minute. What is his speed at 4.5 minutes? Indicate units in the answer.

(c) What is Danh's speed when he is riding the fastest? Include units in the answer.

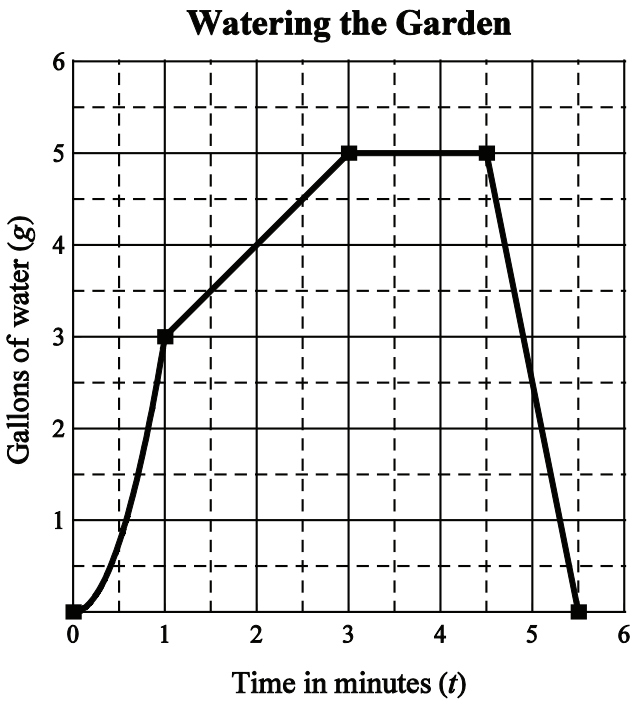
(d) At what three times is Danh riding 0.1 miles per minute? Two of the three times can be identified from the graph. To determine the third time exactly, use the equation, $s = 0.3t - 1.5$, where s is the speed in miles per minute and t is the time in minutes. Show the work that leads to your answer for determining the third time.

(e) Since the area between the graph and the time axis represents the distance that Danh travels, how far does Danh ride during the first 4 minutes? Include units in the answer.

Algebra 1 2011 Free Response Question - Calculator Allowed

Daisy is preparing to water the flowers in her garden. She places a 6-gallon watering can under the outside faucet and gradually turns on the water so that the rate at which the water is flowing into the watering can is increasing. At one minute, she adjusts the water flow to a steady rate. When the watering can contains 5 gallons of water, she removes it from under the faucet and carries it to the garden where she pours the water along the row of flowers.

The graph and the table show the amount of water in the watering can at any given time.



t	$g(t)$
0	0
$\frac{1}{2}$	$\frac{3}{4}$
1	3
3	5
$4\frac{1}{2}$	5
$5\frac{1}{2}$	0

(a) At $2\frac{1}{2}$ minutes, how much water is in the watering can? Include units in your answer.

(b) According to the graph, for how long does the watering can contain the maximum amount of water added from the faucet? Include units in your answer.

(c) How fast is the water flowing into the watering can between 1 minute and 3 minutes? In other words, what is the change in the number of gallons of water divided by the change in the number of minutes? Show the work that leads to the answer and include units.

Between $\frac{1}{2}$ minute and 1 minute, what is the average rate of change in the number of gallons of water in the watering can with respect to time? Show the work that leads to the answer and include units.

What is the numerical difference between the rates for the two time periods? State the answer as a decimal or as a fraction in simplest form and include units in the answer.

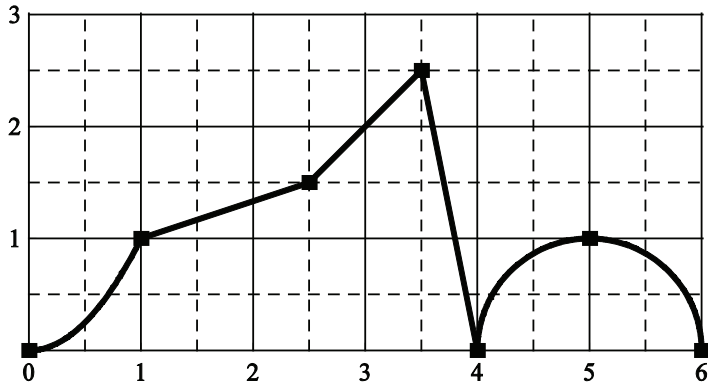
(d) What is the equation, $g(t)$, for the number of gallons of water in the watering can?

$$g(t) = \begin{cases} 3t^2, & 0 \le t \le 1 \\ \underline{\qquad\qquad}, & 1 \le t \le 3 \\ \underline{\qquad\qquad}, & 3 \le t \le 4.5 \\ \underline{\qquad\qquad}, & 4.5 \le t \le 5.5 \end{cases}$$

(e) When Daisy is pouring the water on the flowers, at what time does the watering can contain exactly 2 gallons of water? Show the work that leads to your answer.

Geometry 2011 Free Response Question - Calculator Allowed

The graph of $g(t)$ consists of a portion of a parabola, three line segments, and a semicircle.



t	$g(t)$
0	0
1	1
$2\frac{1}{2}$	$1\frac{1}{2}$
$3\frac{1}{2}$	$2\frac{1}{2}$
4	0
5	1
6	0

(a) The quadrilateral bounded by the graph of $g(t)$ and the t-axis on the interval $2\frac{1}{2} \le t \le 4$ can be divided into two geometric shapes. What is the equation of the vertical line that divides the quadrilateral into a right trapezoid and a right triangle?

What is the area bounded by the graph of $g(t)$ and the t-axis on the interval $2\frac{1}{2} \le t \le 4$? Show the work that leads to the answer.

(b) Fill in the blanks to complete the equation for the piecewise function, $g(t)$.

$$g(t)=\begin{cases} t^2, & 0\le t\le 1 \\ \frac{1}{3}(t-1)+1, & 1\le t\le 2.5 \\ (t-3)+2, & 2.5\le t\le 3.5 \\ \underline{\qquad\qquad}, & 3.5\le t\le 4 \\ \sqrt{1-\left(t-\underline{\quad}\right)^2}, & 4\le t\le 6 \end{cases}$$

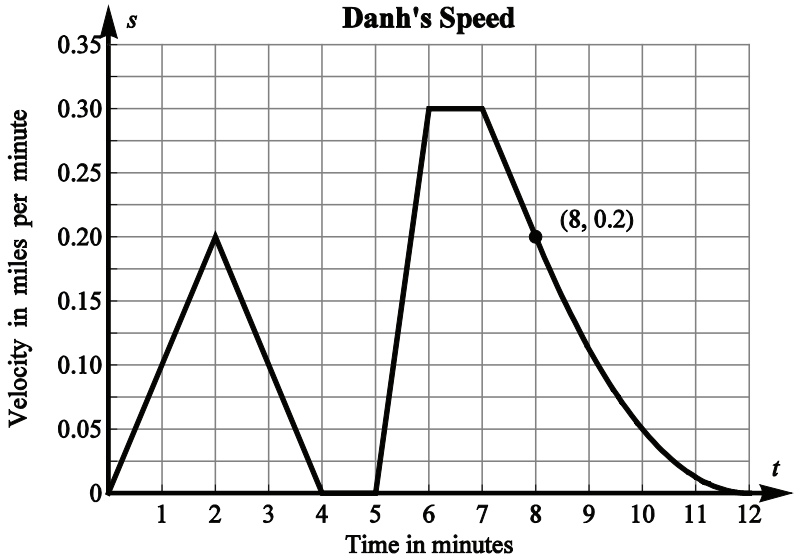
(c) What is the value of t the first time that $g(t)=1.375$?

(d) What is the value of $g(t)$ when $t=5.5$?

(e) For what three non-adjacent intervals does the average rate of change of g with respect to t have a value of one?

Algebra 2 2010 Free Response Question – Calculator Not Allowed

Danh rides his bicycle along a straight road starting from home at $t = 0$ minutes. Danh's velocity, in miles per minute, is modeled by the graph of the piecewise function provided. For the first eight minutes, the velocity graph consists of six linear segments. For $8 \le t \le 12$ minutes, the velocity graph is parabolic with a vertex at (12, 0).



t	$v(t)$
0	0
2	0.2
4	0
5	0
6	0.3
7	0.3
8	0.2
12	0

(a) Write a piecewise defined equation for the velocity function.

$$v(t) = \begin{cases} \underline{\qquad\qquad\qquad}, & 0 \le t \le 2 \\ \underline{\qquad\qquad\qquad}, & 2 \le t \le 4 \\ 0, & 4 \le t \le 5 \\ \underline{\qquad\qquad\qquad}, & 5 \le t \le 6 \\ 0.3, & 6 \le t \le 7 \\ \underline{\qquad\qquad\qquad}, & 7 \le t \le 8 \\ \underline{\qquad\qquad\qquad}, & 8 \le t \le 12 \end{cases}$$

(b) Danh's velocity is 0.25 miles per minute at 7.5 minutes. At what other time is Danh's velocity 0.25 miles per minute? Show the work that leads to your answer.

(c) The area between the velocity function and the t-axis represents the distance that Danh travels. If he rides approximately 0.267 miles between 8 and 12 minutes, how far does Danh ride between 7 and 12 minutes? Show the work that leads to your answer and give the answer to the nearest tenth of a mile.

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Appendix

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Standards for Mathematical Practice

MP.1 - Make sense of problems and persevere in solving them.
Mathematically proficient students:

- start by explaining to themselves the meaning of a problem and looking for entry points to its solution.
- analyze givens, constraints, relationships, and goals.
- make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt.
- consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution.
- monitor and evaluate their progress and change course if necessary.
- Older students might, depending on the context of the problem,
    - transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need,
    - explain correspondences between equations, verbal descriptions, tables, and graphs,
    - draw diagrams of important features and relationships, graph data, and search for regularity or trends.
- Younger students might:
    - rely on using concrete objects or pictures to help conceptualize and solve a problem.
- check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?"
- understand the approaches of others to solving complex problems and identify correspondences between different approaches.

In assessments, the question:

- is designed to take a typical student a long time to solve.
- leads to a more difficult problem.
- requires a large number of routine and fairly easy steps.
- contains several "givens."
- the statement of the problem itself is designed not to allow for jumping in and working the problem immediately.
- posed using abstract statements that must be parsed carefully before they make sense.
- require students to construct their own solution pathway rather than to follow a provided one.
- may be unscaffolded so that a multi-step strategy must be autonomously devised by the student.
- involve ideas that are currently at the forefront of the student's developing mathematical knowledge in a word problem.

MP.2 - Reason abstractly and quantitatively.
Mathematically proficient students:

- make sense of quantities and their relationships in problem situations.
- bring two complementary abilities to bear on problems involving quantitative relationships:
  - the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and
  - the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.
- use quantitative reasoning that entails habits of creating a coherent representation of the problem at hand:
  - considering the units involved;
  - attending to the meaning of quantities, not just how to compute them; and
  - knowing and flexibly using different properties of operations and objects.

In assessment, the question is designed to:

- be contextual so that the student can gain insight into the problem by relating the algebraic form of an answer or intermediate step to the given context.
- require the use symbolic calculations to generalize a situation and draw conclusions from those calculations.

MP.3 - Construct viable arguments and critique the reasoning of others.
Mathematically proficient students:

- understand and use stated assumptions, definitions, and previously established results in constructing arguments.
- make conjectures and build a logical progression of statements to explore the truth of their conjectures.
- analyze situations by breaking them into cases, and can recognize and use counterexamples.
- justify their conclusions, communicate them to others, and respond to the arguments of others.
- reason inductively about data, making plausible arguments that take into account the context from which the data arose.
- compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and – if there is a flaw in an argument – explain what it is.
    - Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades.
    - Later, students learn to determine domains to which an argument applies.
- listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

In assessments, require students to:

- base explanations/reasoning on concrete referents such as diagrams (whether provided in the prompt or constructed by the student).
- construct, autonomously, chains of reasoning that will justify or refute propositions or conjectures.
- determine conditions under which an argument does and does not apply.
- distinguish correct explanations/reasoning from that which is flawed, and – if there is a flaw in the argument – explain what it is.
- provide informal justifications.
- use of diagrams, words, and/or equations to solve.
- reason about key grade-level mathematics.
- apply rigorous deductive proof based on clearly stated axioms.
- state logical assumptions being used.
- test propositions or conjectures with specific examples.
- apply a series of logical and well-motivated steps with precise language and terms.

MP.4 - Model with mathematics.
Mathematically proficient students:

- apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.
  - In early grades, this might be as simple as writing an addition equation to describe a situation.
  - In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.
  - By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another.
- apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.
- identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
- analyze those relationships mathematically to draw conclusions.
- routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

In assessments, require students to:

- apply a known technique from pure mathematics to a real-world situation in which the technique yields valuable results even though it is not obviously applicable in a strict mathematical sense.
- execute some or all of the modeling cycle: formulate, compute, interpret, validate, and report.
- select from a data source, analyze the data and draw reasonable conclusions from it, often resulting in an evaluation or recommendation.
- use reasonable estimates of known quantities in a chain of reasoning that yields an estimate of an unknown quantity.
- make assumptions and simplifications.
- select from the data at hand or estimate data that are missing.
- use reasonable estimates of known quantities in a chain of reasoning that yields an estimate of an unknown quantity.

MP.5 - Use appropriate tools strategically.
Mathematically proficient students:

- consider the available tools when solving a mathematical problem (these tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software).
- are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations.
- High school students:
  - analyze graphs of functions and solutions generated using a graphing calculator.
  - detect possible errors by strategically using estimation and other mathematical knowledge.
  - when making mathematical models, know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data.
- identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.
- use technological tools to explore and deepen their understanding of concepts.

In assessments, questions involve

- making the coordinate plane essential for solving the problem, yet no direction is given to the student to use coordinates.
- creating circumstances for poor use or misuse of tools.
- posing questions that are fairly easy to solve or to answer correctly if a diagram is drawn first, but very hard to solve or to answer correctly if a diagram is not drawn, yet no direction is given to draw a diagram.
- using formulas or conversions where there is no prompting to use them.
- data sets of 15-30 numbers.
- using a calculator to test conjectures with many specific cases.
- substituting messy numerical values into a complicated expression and find the numerical result.

MP.6 - Attend to precision.
Mathematically proficient students:

- try to communicate precisely to others.
- try to use clear definitions in discussion with others and in their own reasoning.
- state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.
- are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem.
- calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.
    - In the elementary grades, students give carefully formulated explanations to each other.
    - By the time they reach high school, students have learned to examine claims and make explicit use of definitions.

In assessments, require students to:

- use reasoned solving of equations, such as those in which extraneous solutions are likely to be found and must be discarded.
- solve algebraic word problems in which success depends on carefully defining variables.
- present solutions to multi-step problems in the form of valid chains of reasoning, using symbols such as equal signs appropriately.

MP.7 - Look for and make use of structure.
Mathematically proficient students:

- look closely to discern a pattern or structure.
  - Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have.
  - Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property.
  - In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$.
  - They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems.
- step back for an overview and shift perspective.
- see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

In assessments, questions:

- can be solved by analyzing parts of figures in relation to one another.
- can be solved by introducing auxiliary lines into a figure.
- reward seeing structure in an algebraic expression and using the structure to rewrite it for a purpose.
- reward or require deferring calculation steps until one sees the overall structure.
- assess how aware students are of how concepts link together and why mathematical procedures work in the way that they do.

MP.8 - Look for and express regularity in repeated reasoning.
Mathematically proficient students:

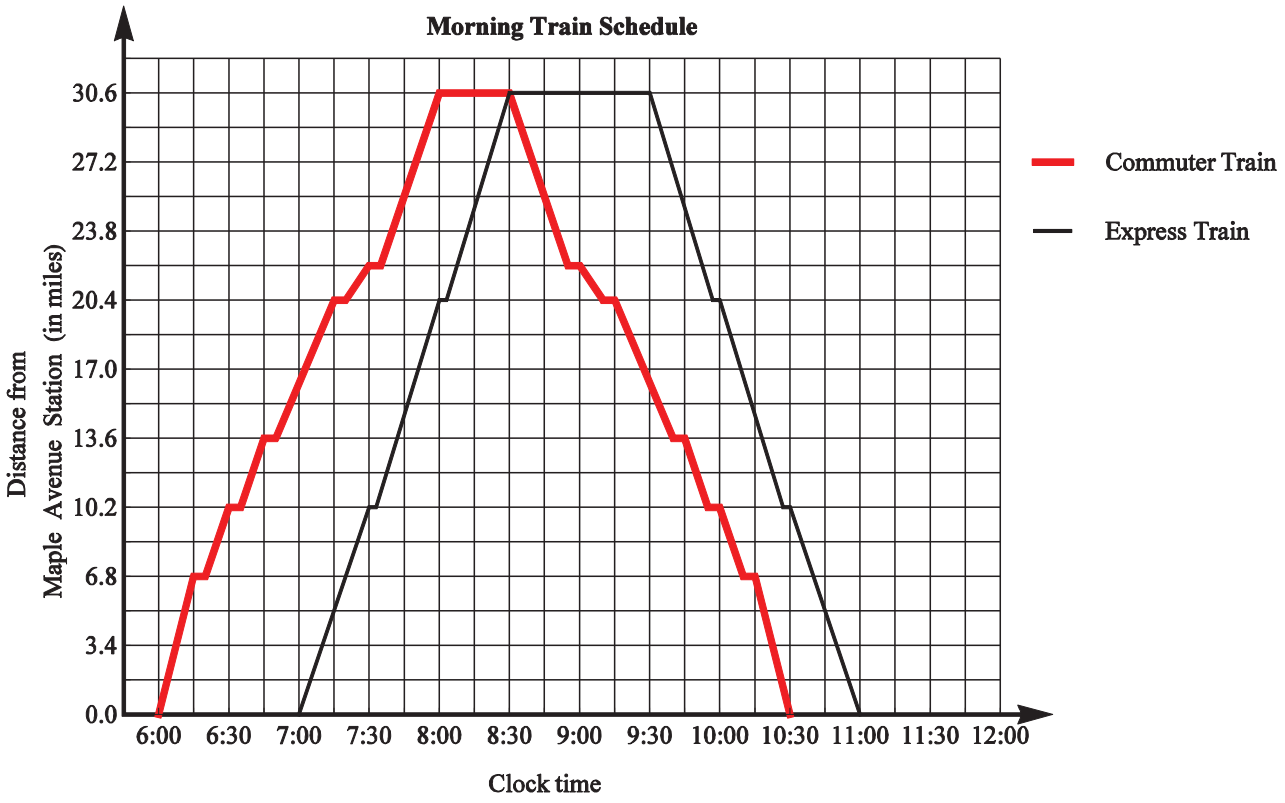
- notice if calculations are repeated, and look both for general methods and for shortcuts.
  - Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal.
  - By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$.
  - Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series.
- maintain oversight of the process, while attending to the details.
- continually evaluate the reasonableness of their intermediate results.

In assessments, questions require:

- repeating calculations to lead to the articulation of a conjecture.
- working repetitively with numerical examples leading without prompting to the writing of equations or functions that describe modeling situations.
- recognizing that tedious and repetitive calculation can be made shorter by observing regularity in the repeated steps.
- answers like "multiplying by any number and then dividing by the same number gets you back to where you started."
- using recursive definitions of functions.
- using patterns to shed light on the addition table, the times table, the properties of operations, the relationship between addition and subtraction or multiplication and division, and the place value system.

Additional Graphs and Materials

All Aboard!

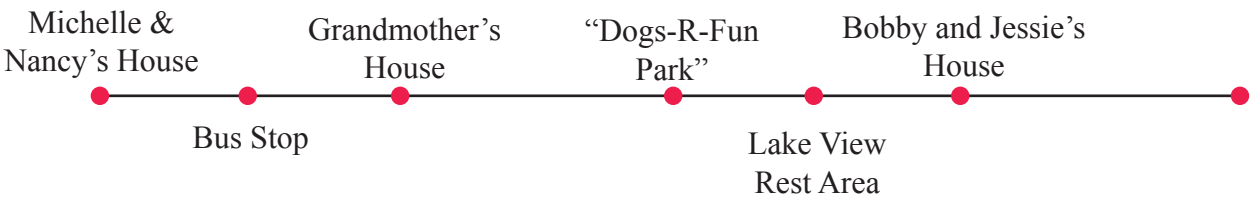


Station Location	Distance from Maple Avenue Station	Commuter Train Stops at Station	Express Train Stops at Station
Ash Lane	6.8 miles	Yes	No
Plum Street	10.2 miles	Yes	Yes
Pine Street	13.6 miles	Yes	No
Pecan Street	20.4 miles	Yes	Yes
Spruce Lane	22.1 miles	Yes	No
Bradford Avenue	30.6 miles	Yes	Yes

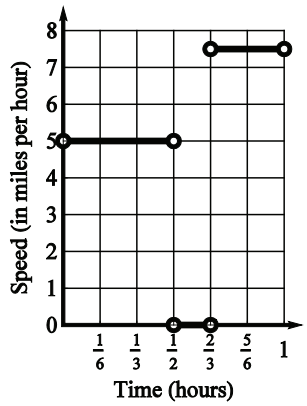
Putting the Pieces Together

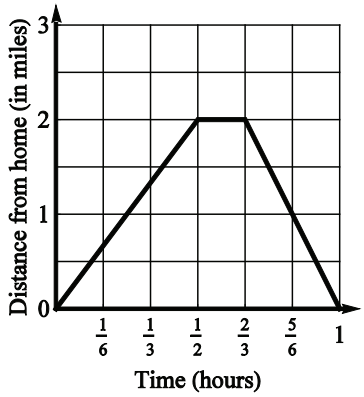
Michelle and Nancy's parents have told them that they must exercise each day after school. They have allotted 1 hour each day for their workouts. This week they decided to describe their workout, create a graph to map their routine and to track their speed, and then write a set of equations for each graph. They like to run, walk, or bike each day in their neighborhood. From Michelle and Nancy's house, the Dogs-R-Fun park is 2 miles away, Uncle Bobby and Aunt Jessie's house is 3 miles away, and Grandmother lives 1 mile away. To ride the bus to school in the morning, they have to walk $\frac{1}{2}$ a mile to the Bus Stop. The Lake View Rest Area is exactly half way between the Dogs-R-Fun Park and Uncle Bobby and Aunt Jessie's House.

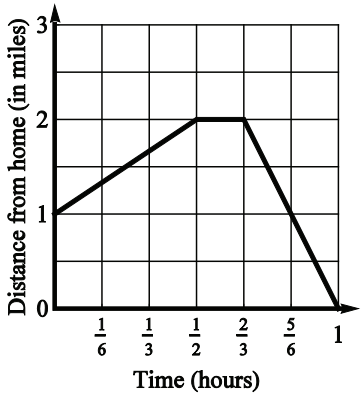
Based upon the verbal description of their exercise for the day, determine the distance-time graph that matches the description and tape it in the appropriate blank and then complete the piecewise-defined function for the distance-time graph. Match the graph of the speed-time based on the distance-time graph, and complete the piecewise function for the speed graph. Fill in the blanks for the description as you work through the activity.

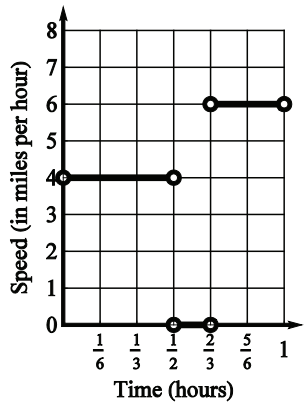


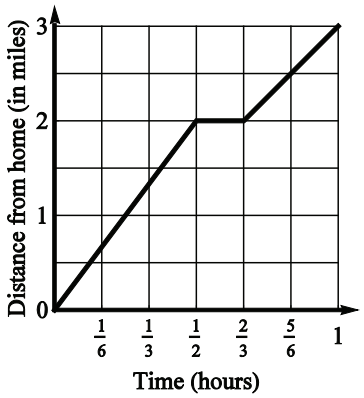
Putting the Pieces Together

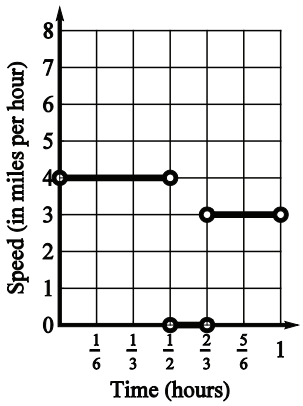


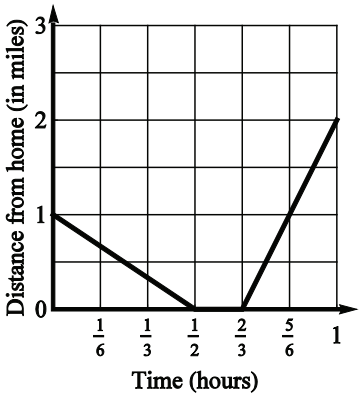


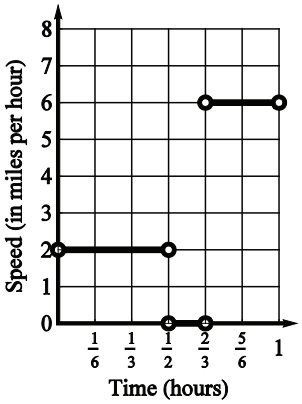


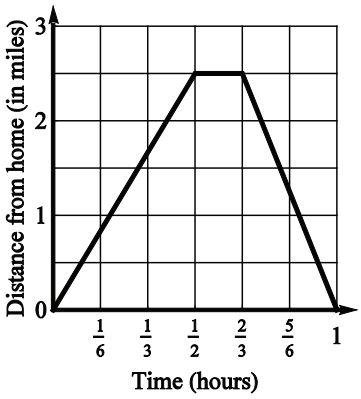


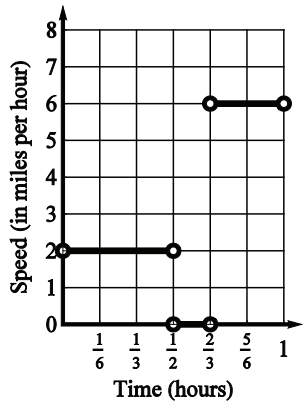




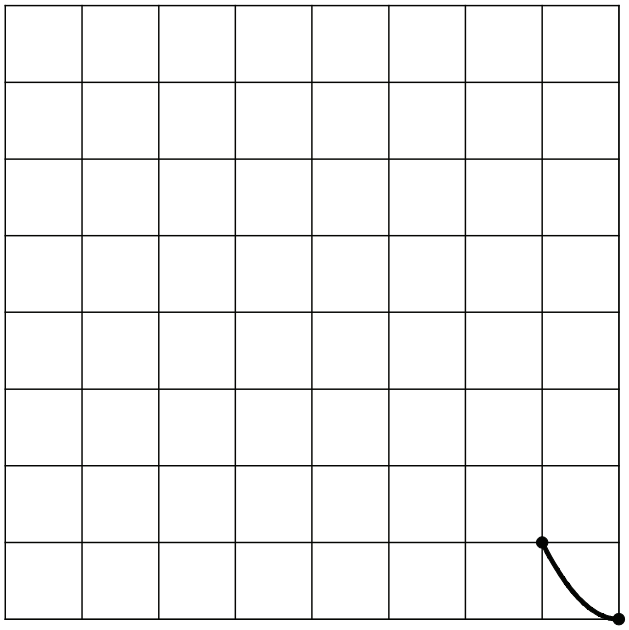


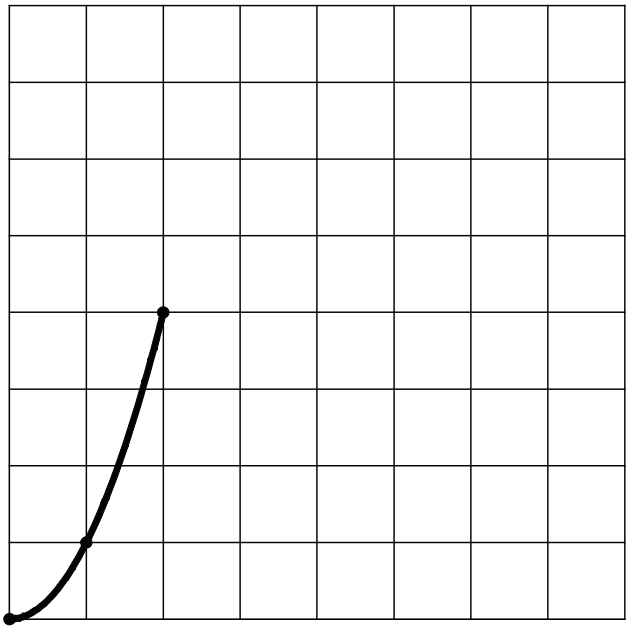


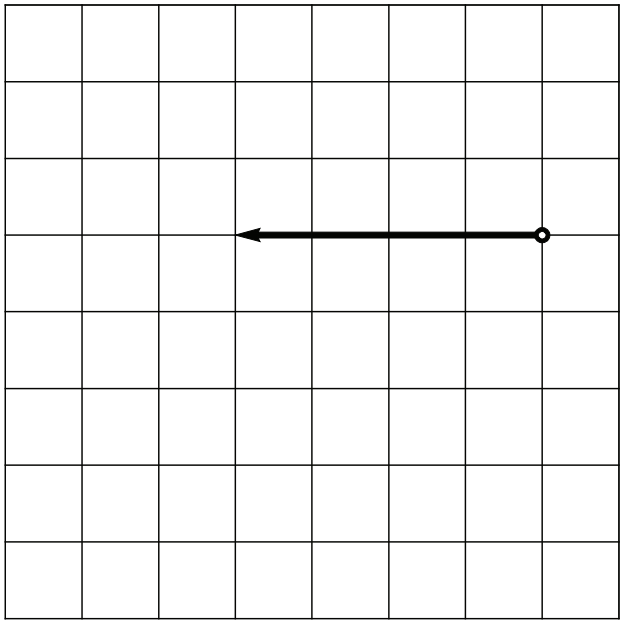


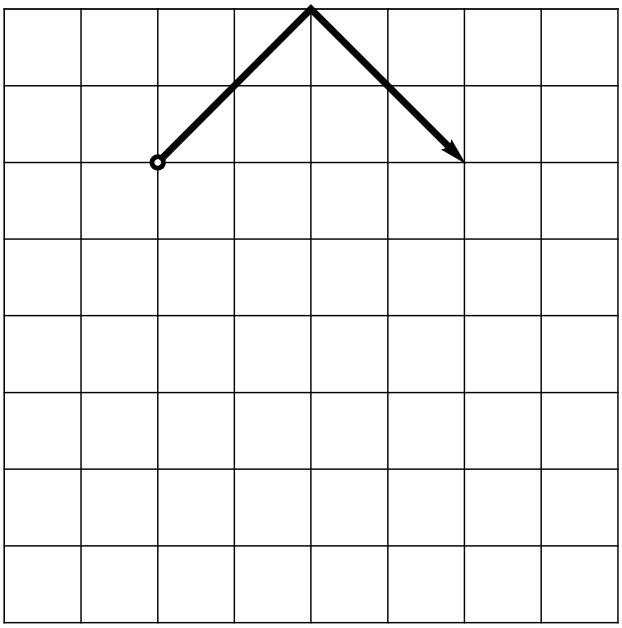


Piecewise Puzzle



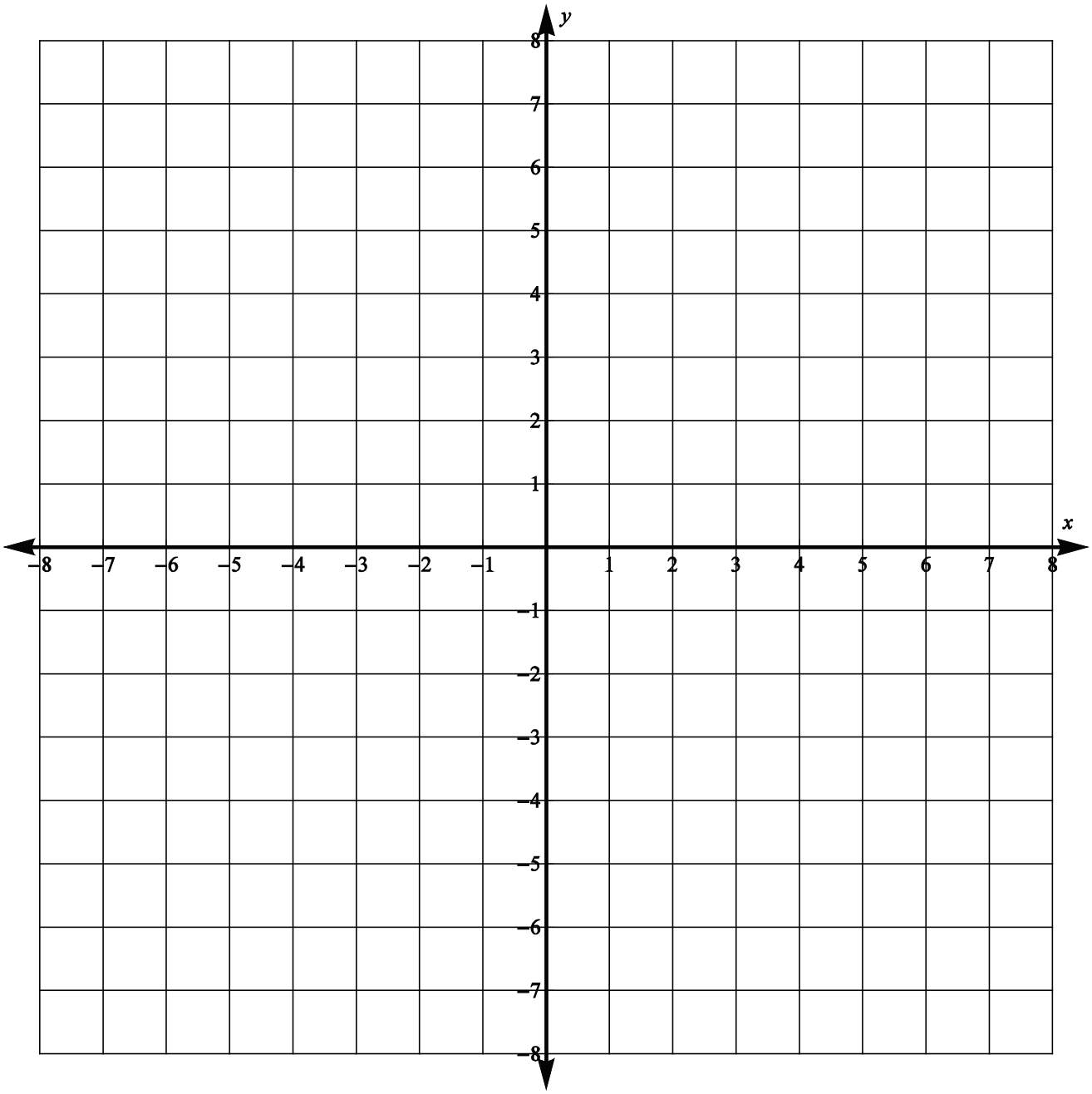






Piecewise Puzzle

COORDINATE MAT



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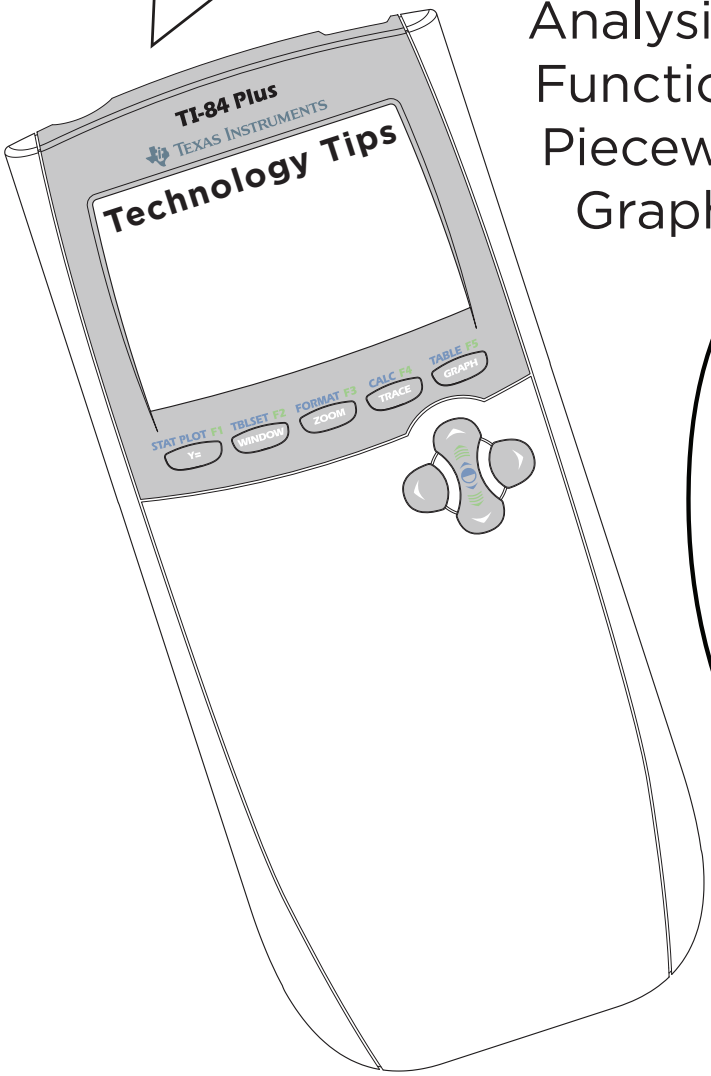
Analysis of Functions: Piecewise Graphs



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